THEOREM OF THE DAY

Fisher's Inequality *If a balanced incomplete block design is specified with parameters* (v, b, r, k, λ) *then* $v \leq b$.

SUM =MMULT(C3:Q12;T3:AC17)

	B0	B1	B2	B3	B4	B5	B6	B7	B8	B9	B10	B11	B12	B13	B14
T0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
T1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	0
T2	1	0	1	0	0	0	1	0	0	0	1	1	1	0	0
T3	1	0	0	1	0	0	0	1	0	0	1	0	0	1	1
T4	0	1	1	0	0	0	0	0	1	0	0	1	0	1	1
T5	0	1	0	0	1	0	0	0	0	1	1	0	1	1	0
T6	0	0	1	0	0	1	0	1	0	1	0	0	1	0	1
T7	0	0	0	1	1	0	1	0	1	0	0	0	1	0	1
T8	0	0	0	1	0	1	1	0	0	1	0	1	0	1	0
T9	0	0	0	0	1	1	0	1	1	0	1	1	0	0	0

x

	T0	T1	T2	T3	T4	T5	T6	T7	T8	T9
B0	1	1	1	1	0	0	0	0	0	0
B1	1	1	0	0	1	1	0	0	0	0
B2	1	0	1	0	1	0	1	0	0	0
B3	1	0	0	1	0	0	0	1	1	0
B4	1	0	0	0	0	1	0	1	0	1
B5	1	0	0	0	0	0	1	0	1	1
B6	0	1	1	0	0	0	0	1	1	0
B7	0	1	0	1	0	0	1	0	0	1
B8	0	1	0	0	1	0	0	1	0	1
B9	0	1	0	0	0	1	1	0	1	
B10	0	0	1	1	0	1	0	0	0	1
B11	0	0	1	0	1	0	0	0	1	1
B12	0	0	1	0	0	1	1	1	0	0
B13	0	0	0	1	1	1	0	0	1	0
B14	0	0	0	1	1	0	1	1	0	0

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	T0	T1	T2	T3	T4	T5	T6	T7	T8	T9
T0	6	2	2	2	2	2	2	2	2	2
T1	2	6	2	2	2	2	2	2	2	2
T2	2	2	6	2	2	2	2	2	2	2
T3	2	2	2	6	2	2	2	2	2	2
T4	2	2	2	2	6	2	2	2	2	2
T5	2	2	2	2	2	6	2	2	2	2
T6	2	2	2	2	2	2	6	2	2	2
T7	2	2	2	2	2	2	2	6	2	2
T8	2	2	2	2	2	2	2	2	6	2
					2	2	2	2	2	6

=MMULT(C3:Q12;T3:AC17)

v	b	r	k	λ
10	15	6	4	2

Row **Ti**, Column **Bj** = 1 if and only if treatment **Ti** is in block **Bj**

In a balanced incomplete block design, or **BIBD**, a set of v **treatments** are selected, with repetition, to form b **blocks**, each being a set of cardinality k, where $k < v$ (whence 'incomplete'), in such a way that

1. every treatment occurs in exactly r blocks ('first order balance'; implies the equality $bk = rv$); and
2. every unordered pair of treatments occurs in exactly λ blocks ('second order balance'; implies $\lambda(v-1) = r(k-1)$ and $r > \lambda$).

A BIBD may be represented by an incidence matrix M, as illustrated above left in the OpenOffice Calc screenshot; if this is multiplied by its transpose M^T (centre) then the balance conditions are represented in the resulting $v \times v$ matrix, MM^T (right), which has r in each diagonal positions and λ everywhere else (encircled we see how the $r = 6$ blocks containing treatment T1 match, exactly $\lambda = 2$ times, those containing treatment T5).

Add to row 1 of MM^T each other row. Subtract column 1 in the resulting matrix from each other column. The result is the matrix shown on the right whose determinant is the product of its diagonal elements, which is $(r+(v-1)\lambda)(r-\lambda)^{v-1}$. Since $r > \lambda$ this is non-zero; in other words $\text{rank}(MM^T) = v$. But $\text{rank}(MM^T) \leq \text{rank}(M) \leq \min(v, b)$. So we must have $\min(v, b) = v$, i.e. $v \leq b$, and this proves Fisher's Inequality. The inequality allows us, for example, to bound block size k, given v and λ: condition 1 above gives $k \leq r$ whence condition 2 gives $\lambda(v-1) \geq k(k-1)$, and now solving for v gives $k \leq \frac{1}{2}(1 + \sqrt{1 + 4\lambda(v-1)})$.

$$\begin{pmatrix} r+(v-1)\lambda & 0 & 0 & \dots & 0 & 0 \\ \lambda & r-\lambda & 0 & \dots & 0 & 0 \\ \lambda & 0 & r-\lambda & 0 & \dots & 0 \\ \vdots & & \vdots & & & \vdots \\ \lambda & 0 & 0 & 0 & \dots & r-\lambda \end{pmatrix}$$

Ronald Fisher's fundamental property of BIBDs dates from 1940. The above proof is due to Raj Chandra Bose (1949).

Web link: btravers.weebly.com/uploads/6/7/2/9/6729909/combinatorial_design_slides_student.pdf.

Further reading: *Combinatorial Designs and Tournaments* by Ian Anderson, OUP, 1997.

Created by Robin Whitty for www.theoremoftheday.org