Nested sums, ballots and Catalan’s triangle

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Catalan's triangle $C(n,k)$

n \ k	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	2						
3	1	3	5	5					
4	1	4	9	14	14				
5	1	5	14	28	42	42			
6	1	6	20	48	90	132	132		
7	1	7	27	75	165	297	429	429	
8	1	8	35	110	275	572	1001	1430	1430

Wikipedia

$C(n,k)$ is the number of sequences of n Xs and k Ys in which the number of Ys never exceeds the number of Xs.

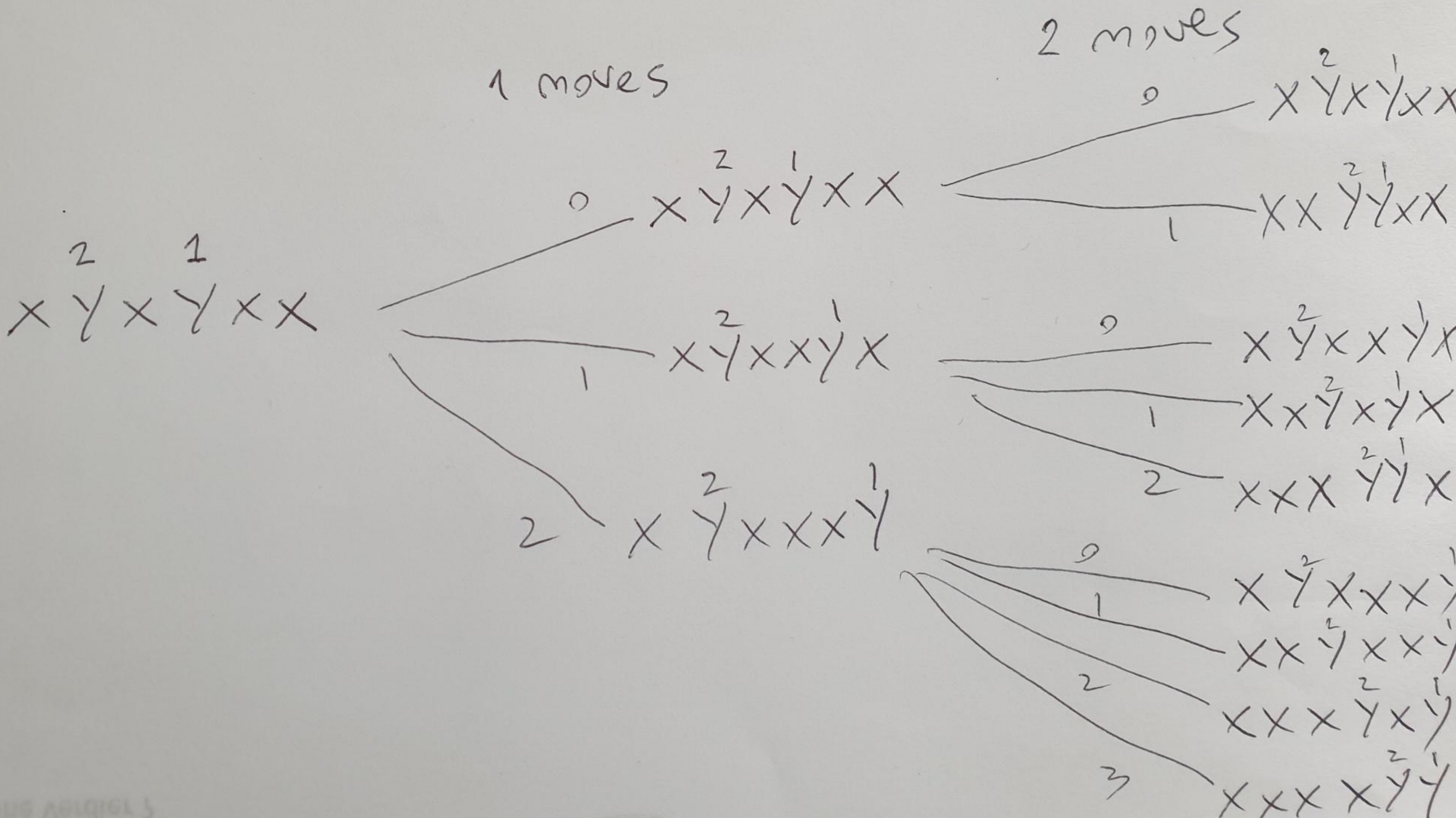
$$c(n,k) = \frac{n+1-k}{n+1}\binom{n+k}{k}$$

$C(4,2)$ = 9

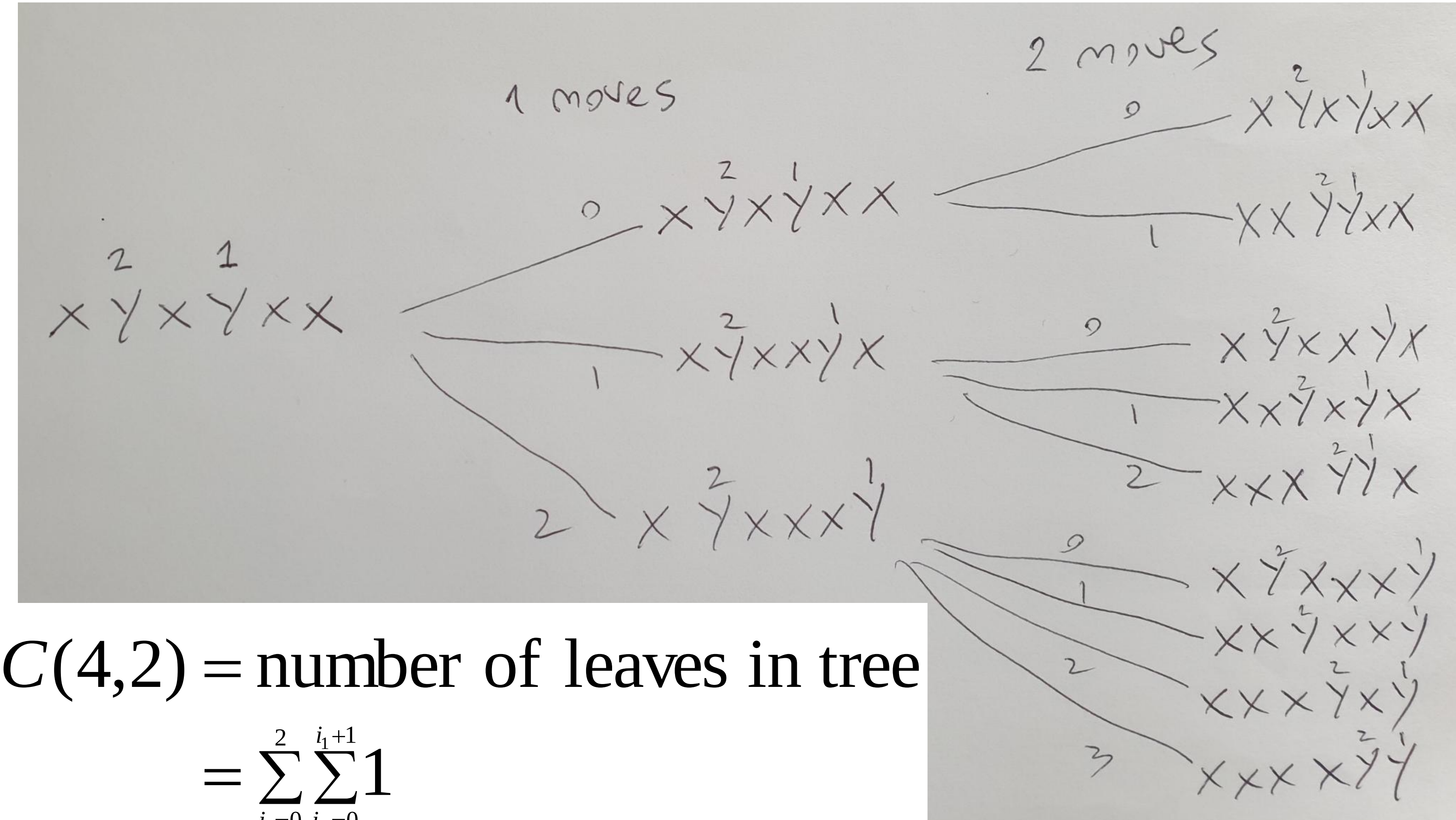
XXXXYY
XXXYXY
XXXYYX
XXYXXY
XXYXYX
XXYYXX
XYXXXY
XYXXYX
XYXYXX

$C(n,k)$ sequences generated from a base sequence

$C(4,2)$ generated from $(XY)^2 X^{4-2}$



$C(n,k)$ as a nested summation



$$C(4,2) = \text{number of leaves in tree}$$

$$= \sum_{i_1=0}^{2} \sum_{i_2=0}^{i_1+1} 1$$

$$C(n,k) = \text{number of leaves in tree with } k \text{ Ys} = \sum_{i_1=0}^{n-k} \sum_{i_2=0}^{i_1+1} \ldots \sum_{i_k=0}^{i_{k-1}+1} 1$$

Computing nested sums recursively

$$S(n,k,t) = \sum_{i_1=0}^{n} \sum_{i_2=0}^{i_1+t} \cdots \sum_{i_k=0}^{i_{k-1}+t} 1$$

$$S(i,j,t) = \begin{cases} 1 & j=0 \\ \sum_{k=0}^{i+t} S(k,j-1,t) & j>0 \end{cases}$$

Evaluating nested sums

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A Note on Nested Sums

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Abstract

We consider several nested sums, and show how binomial coefficients, Stirling numbers of the second kind and Gaussian binomial coefficients can be written as nested sums. We use this to find the rate of growth for diagonals of Stirling numbers of the second kind, as well as another proof of a known identity for Gaussian binomial coefficients.

1 Introduction

In this note we are interested in looking at nested sums and finding simple closed form expressions for them. Our motivation for looking at these sums was the following identity.

$$\sum_{k_p=0}^{n} \sum_{k_{p-1}=0}^{k_p} \cdots \sum_{k_1=0}^{k_2} 1 = \binom{n+p}{n} \qquad (1)$$

Butler and Karasik's identity

$$\sum_{k_p=0}^{n} \sum_{k_{p-1}=0}^{k_p} \cdots \sum_{k_1=0}^{k_2} 1 = \binom{n+p}{n} \qquad (1)$$

One way to see this is to note that on the left hand side we count the number of occurrences of $n \geq k_p \geq k_{p-1} \geq \cdots \geq k_1 \geq 0$. There is a one-to-one correspondence between these and nonnegative $(p+1)$-tuples which sum to n, i.e., $(n-k_p, k_p-k_{p-1}, \ldots, k_1-0)$. The number of such tuples is the same as the number of ways to put n identical balls into $p+1$ distinct bins. This can be counted by arranging the n balls and p dividers between bins in a row which can be done in $\binom{n+p}{n}$, the right hand side.

And now:

$$\sum_{k_p=0}^{n} \sum_{k_{p-1}=0}^{k_p+1} \cdots \sum_{k_1=0}^{k_2+1} 1 = ?$$

A Catalan recurrence

To

$$S(n,k,t) = \sum_{i_1=0}^{n} \sum_{i_2=0}^{i_1+t} \cdots \sum_{i_k=0}^{i_{k-1}+t} 1$$

Then $$S(n,k,1) = \sum_{i=0}^{k-1} T(k-1,i)\, S(n,k-i,0)$$

where $$T(n,m) = \binom{n+m}{n} \frac{n-m+1}{n+1} \; : \text{Catalan's } \Delta.$$

E.g. $$S(N,3,1) = \underset{\substack{\uparrow \\ T(2,0)}}{1} . S(N,3,0) + \underset{\substack{\uparrow \\ T(2,1)}}{2}\, S(N,2,0) + \underset{\substack{\uparrow \\ T(2,2)}}{2}\, S(N,1,0)$$

n \ *k*	0	1	2
0	1		
1	1	1	
2	1	2	2