

# Rook polynomials on 2-dimensional surfaces and graceful graphs

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## Abstract

By a simple trick we may generalise the rook polynomial for an  $n \times n$  chessboard to various 2-dimensional surfaces, the conventional chessboard corresponding to the torus. In the case of the Möbius band and the Klein bottle, there is a close connection to graceful labellings of graphs.

## 1 Introduction

Let  $S_n$  denote the group of permutations of  $[n] = \{1, \dots, n\}$ . For a square matrix  $A = [a_{ij}]$  of order  $n$ , the *permanent* of  $A$ , denoted  $\text{per } A$ , is defined by

$$\text{per } A = \sum_{\alpha \in S_n} \prod_{i=1}^n a_{i\alpha(i)}.$$

For an  $m \times n$  matrix  $A = [a_{ij}]$ , with  $m \leq n$ , and for subsets of  $X$  and  $Y$  of  $[m]$  and  $[n]$ , respectively, we denote by  $A[X|Y]$  the submatrix of  $A$  consisting of the rows indexed by  $X$  and the columns indexed by  $Y$ . Then we extend the definition of the permanent to the non-square matrix  $A$ , by

$$\text{per } A = \sum_{X \subseteq [n], |X|=m} \text{per } A[[m]|X].$$

Following [1], we define

$$\sigma_k(A) = \begin{cases} 1 & k = 0 \\ 0 & k > m \\ \sum_{X,Y} \text{per } A[X|Y] & 1 \leq k \leq m \end{cases}$$

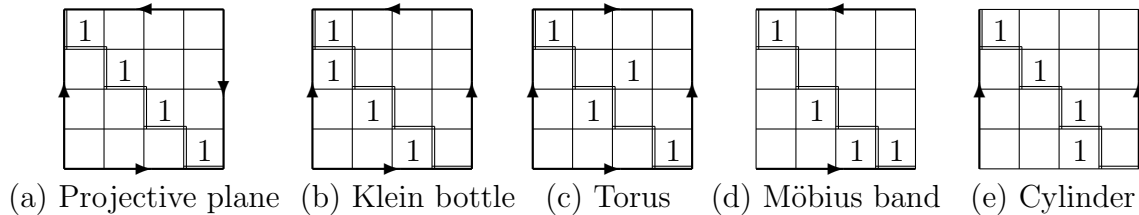


Figure 1: chessboards on two-dimensional surfaces.

where the summation runs over  $k$ -subsets of  $X$  of  $[m]$  and  $k$ -subsets  $Y$  of  $[n]$ .

If  $A$  is a  $(0, 1)$ -matrix then it may be identified with an  $m \times n$  chessboard in which squares corresponding to zeros in  $A$  have been ‘deleted’, and upon which we are to place rooks, which ‘attack’ along the row and column of the square they inhabit. Then  $\sigma_k(A)$  specifies the number of ways that  $k$  mutually non-attacking rooks may be placed on the chessboard. In particular,  $\sigma_m(A) = \text{per } A$  is the number of ways to place a rook on each row of the chessboard, with no rooks attacking each other. The vector  $r_A = [\sigma_n(A), \dots, \sigma_0(A)]$  is called the *rook vector* of  $A$  and the polynomial  $r(A; x) = \sum \sigma_i(A)x^i$  is the *rook polynomial* of  $A$ . If no squares are deleted then  $A$  is the  $m \times n$  all-ones matrix which we denote  $J_{m,n}$ , or  $J_n$  in the case where  $m = n$ .

In this paper we shall largely confine ourselves to  $n \times n$  chessboards with no deleted squares but we shall generalise in a different way by identifying pairs of opposite edges of the chessboard in a manner familiar from the topology of two-dimensional surfaces. At the same time we draw a boundary line down the diagonal of the chessboard and decree that rooks cannot attack each across this boundary. Fig. 1 shows the five possible surfaces: in fig. 1(a), the projective plane is shown with the only possible legal placement of four rooks. This placement is also legal for each of the following four boards. Fig. 1(b) shows a placement which is legal for the Klein bottle, Möbius band and cylinder. Fig. 1(c) shows a placement which is legal only for the torus and cylinder. Fig. 1(d) shows a placement which is legal only for the Möbius band; while fig. 1(e) shows a placement which is legal only for the cylinder.

It is immediately apparent that the rook polynomial of the torus is equivalent to the classical definition of the rook polynomial, since the identification of the sides of the chessboard render the diagonal barrier irrelevant. The rook polynomial in this case, deriving from the matrix  $J_n$ , is well-known:

$$r(J_n; x) = \sum_{k=0}^n k! \binom{n}{k}^2 x^k \quad (1)$$

We further observe that the rook polynomials for the cylinder and Möbius band are identical, that is the boards are *rook-equivalent*, since both chessboards may be unfolded into a ‘pyramid’ form, illustrated in fig. 2 for the case  $n = 4$ .

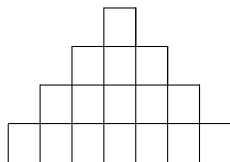


Figure 2: conventional pyramid chessboard equivalent to the Möbius band and cylinder in fig. 1(d) and (e).

Our attention in this paper is first given to showing that the rook polynomial for this pyramid chessboard, with base  $2n - 1$  is identical to the classical  $n \times n$  rook polynomial, given in (1). Thus the three final boards in fig. 1: the torus, the Möbius band and the cylinder, are all rook-equivalent. We then turn our attention to the Klein bottle; we have not yet been able to calculate the rook polynomial in this case but, by exploiting a connection with graceful graphs, we are at least able to give the leading term, expressed as a permanent evaluation. The projective plane has received no attention at all, so far as we know.

## 2 The torus, Möbius band and cylinder

The pyramid chessboard representing the Möbius band and cylinder, illustrated in fig. 2, corresponds to an  $n \times 2n - 1$   $(0, 1)$ -matrix  $P_n = [p_{ij}]$  defined by

$$p_{ij} = \begin{cases} 1 & i > \max(n - j, j - n) \\ 0 & \text{otherwise} \end{cases} .$$

It is immediate, by column permutations, that the rook vector of  $P_n$  is the same as the rook vector of the  $n \times 2n - 1$  Ferrers matrix  $F_{2n-1}(b_1, \dots, b_n)$ , where  $b_i = 2i - 1, i = 1, \dots, n$ , the  $i$ th row of this matrix consisting of  $b_i$  ones, followed by  $2n - 1 - b_i$  zeros. Cheon et al [1] give a convenient method for calculating rook vectors of Ferrers matrices and we will use this calculation to confirm that the rook polynomials for the torus, Möbius band and cylinder are the same.

For an  $n \times n$  Ferrers matrix  $A = F_n(b_1, \dots, b_n)$ , write

$$f_A(k) = \prod_{i=1}^n (k + b_i - i + 1), \quad k = 0, \dots, n. \quad (2)$$

Let  $\Delta_n$  be the  $(n+1) \times (n+1)$  matrix of binomial coefficients:

$$\Delta_n = \begin{bmatrix} \binom{0}{0} & 0 & 0 & \cdots & 0 \\ \binom{1}{0} & \binom{1}{1} & 0 & \cdots & 0 \\ \binom{2}{0} & \binom{2}{1} & \binom{2}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n} \end{bmatrix},$$

and let  $D_n = \text{diag}(0!, 1!, \dots, n!)$ . Then

**Theorem 1** ([1]) *The rook vector  $r_A$  of the Ferrers matrix  $A = F_n(b_1, \dots, b_n)$  is given by*

$$r_A = D_n^{-1} \Delta_n^{-1} (f_A(0), \dots, f_A(n))^T.$$

It is observed, in [1], that  $\Delta_n^{-1}$  is given by

$$\Delta_n^{-1} = \begin{bmatrix} \binom{0}{0} & 0 & 0 & \cdots & 0 \\ -\binom{1}{0} & \binom{1}{1} & 0 & \cdots & 0 \\ \binom{2}{0} & -\binom{2}{1} & \binom{2}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ (-1)^n \binom{n}{0} & (-1)^{n+1} \binom{n}{1} & (-1)^{n+2} \binom{n}{2} & \cdots & \binom{n}{n} \end{bmatrix}. \quad (3)$$

If the  $n \times 2n-1$  matrix  $F_{2n-1}(b_1, \dots, b_n)$  is extended to a  $2n-1 \times 2n-1$  matrix  $A$  by adding  $n-1$  rows of zeros then it can be calculated from (2) that

$$f_A(k) = \begin{cases} 0 & k = 0, \dots, n-2 \\ \frac{k!(k+1)!}{(k-n+1)!^2} & k = n-1, \dots, 2n-1 \end{cases} \quad (4)$$

Now theorem 1, together with (3) and (4) gives

$$\begin{aligned} r_{n-k}(A) &= \frac{1}{(n+k-1)!} \sum_{i=0}^k (-1)^{i+k} \binom{n+k-1}{k-i} \frac{(n+i-1)!(n+i)!}{(i!)^2} \\ &= \sum_{i=0}^k \frac{(-1)^{i+k}}{i!k!} \binom{k}{i} (n+i)! \end{aligned} \quad (5)$$

We now calculate the rook vector of the traditional chessboard (the torus) another way and find it is identical to (5). Let  $O_{p,q}$  denote the  $p \times q$  zero matrix. We extend  $A$  to an  $n \times n$  matrix  $\tilde{A}$  by adding  $n-m$  rows of zeros:

$$\tilde{A} = \begin{bmatrix} A \\ O_{n-m,n} \end{bmatrix},$$

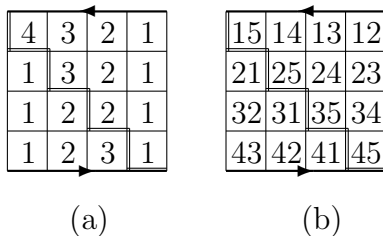


Figure 3: Möbius band with column partitions labelled (a) by integers, (b) by differences between pair ordinates.

and define matrices  $Y_k(A)$  by  $Y_0(A) = \tilde{A}$  and

$$Y_k(A) = \begin{bmatrix} \tilde{A} & J_{n,k} \\ J_{k,n} & J_k \end{bmatrix}, \quad k = 1, \dots, n.$$

Let  $y_k(A)$  denote  $\text{per } Y_k(A)$ ,  $k = 0, \dots, n$ . Then

**Theorem 2** ([1]) *For an  $m \times n$  matrix  $A$ , we have*

$$\sigma_{n-k}(A) = \sum_{i=0}^k \frac{(-1)^{i+k}}{i!k!} \binom{k}{i} y_i(A), \quad k = 0, \dots, n.$$

Comparing this to (5) it remains to observe that, for the traditional  $n \times n$  chessboard,  $A = J_n$  and  $y_i(A) = (n+i)!$  and we have

**Theorem 3** *The torus, Möbius band and cylinder are all rook-equivalent.*

There is a simple bijection between placements of  $n$  rooks on the torus and on the Möbius band: observe in fig. 1(d) that the attacking lines of the rooks along columns partition the columns into  $n$  sets as illustrated in fig. 3(a). It is clear that a rook placement will involve placing one rook in each column partition set. There is exactly one choice in the set labelled  $n$ ; this leaves two choices out of the three for set  $n-1$ ; now we have three choices out of the four for set  $n-2$ , and so on. In this way we generate exactly  $n!$  rook placements, in one-to-one correspondence with the  $n!$  placements of  $n$  rooks on the torus. We have been unable to extend this correspondence to answer the following:

**Question 1** *Find a bijection between the placements of  $n-k$  rooks on the torus and Möbius band, for  $k < n$  (thus giving a bijective proof of theorem 3.)*

Another way of looking at the placement of  $n$  rooks is to label the cells of the Möbius band with a coordinate pair in such a way that the difference between the

15	14	13	12
21	25	24	23
32	31	35	34
43	42	41	45

Figure 4: Klein bottle with coordinates.

ordinates specifies the column partition set, as shown in fig. 3(b). In this case we see that each placement of  $n$  rooks corresponds to a graceful graph, that is, a graph on  $n$  edges in which vertices are labelled 1 to  $n + 1$  and each edge is labelled with the difference of its end nodes, in such a way that the edges are labelled 1 to  $n$ . The rook placements on the Möbius band demonstrate that there are exactly  $n!$  graceful graphs on  $n$  edges, a fact first observed by Sheppard [3].

### 3 The Klein bottle and graceful labellings

We continue with the theme of graceful labellings by transferring the coordinates of fig. 3(b) to the Klein bottle, as shown in fig. 4. Whereas placing  $n$  rooks on the  $n \times n$  Möbius band generates all graceful graphs, the Klein bottle restricts attention to spanning subgraphs of the complete graph on  $n + 1$  vertices because one rook is required in each row, so that there is an edge incident with each vertex (vertex  $n + 1$  is incident with the edge  $(1, n + 1)$  by virtue of the fact that the top left-hand cell must contain a rook.)

Spanning subgraphs of the complete graph  $K_{n+1}$  having  $n$  edges either contain a cycle or are spanning trees. In fact, it is clear that, as in fig. 4, the coordinates in each position may be interpreted as directed edges, so that a rook placement corresponds to a spanning subgraph of  $K_{n+1}$ , oriented so that each vertex, other than  $n + 1$ , has outdegree one and the edges not contained in cycles are directed into vertex  $n + 1$ . If the subgraph has a cycle then there will be two corresponding rook placements, one for each orientation of the cycle. The smallest examples occur when  $n + 1 = 7$  in which there are two labelled subgraphs containing cycles, giving rise to four different rook placements. Two are shown in fig. 5.

Spanning trees are enumerated as a determinant evaluation in the so-called Matrix Tree Theorem and we shall adapt this to enumerate graceful trees and  $n$ -rook placements on the  $n \times n$  Klein bottle. For an oriented graph  $G = (V, E)$ , with  $|V| = m$  and  $|E| = n$ , let the incidence matrix  $B(G) = [b_{ij}]$  denote the  $m \times n$  matrix whose rows and columns are indexed by  $V$  and  $E$ , respectively with the entry

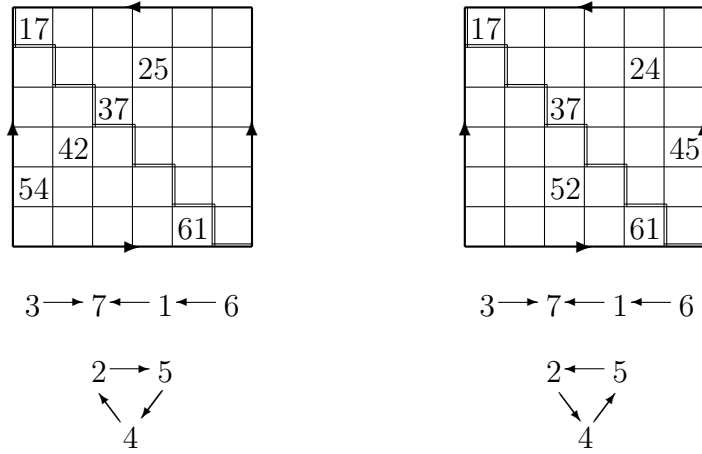


Figure 5: two rook placements corresponding to different orientations of the same graph.

corresponding to vertex  $i$  and edge  $j$  being given by

$$b_{ij} = \begin{cases} 1 & \text{if edge } j \text{ is directed away from vertex } i \\ -1 & \text{if edge } j \text{ is directed towards vertex } i \\ 0 & \text{if edge } j \text{ is not incident with vertex } i \end{cases}$$

It is well-known that, given an  $m - 1 \times m - 1$  submatrix  $X$  of  $B(G)$ , with the columns of  $X$  corresponding to subset  $F \subseteq E$ , the determinant of  $X$  satisfies

$$\det X = \begin{cases} \pm 1 & \text{if the edges in } F \text{ form a tree} \\ 0 & \text{otherwise.} \end{cases}$$

Let  $B_i$  and  $B_j$  be copies of  $B(G)$  with the  $i$ th and  $j$ th row deleted respectively and consider the product  $B_i B_j^T$ . We will apply the Cauchy-Binet theorem: for any two matrices,  $A$ ,  $m \times n$ , and  $B$ ,  $n \times m$ , with  $m \leq n$ ,

$$\det AB = \sum_{X \subseteq [n], |X|=m} \det A[[m]|X] \det B[X|[m]].$$

For  $B_i$  and  $B_j$ , the sum will count  $\pm 1$  for each spanning tree of  $G$  and zero for all other submatrices. With a little care over the signs we have:

**Theorem 4 (The Matrix Tree Theorem)** *Any row and column deleted minor of  $B(G)B(G)^T$  has determinant whose absolute value equals the number of spanning trees of  $G$ .*

We shall enumerate graceful trees as a subset of the spanning trees of the complete graph, which for notational convenience we shall now take on  $n$  vertices (so

we shall eventually deal with  $n - 1$ -rook placements.) Let  $B_n$  be the  $n \times \frac{1}{2}n(n - 1)$  incidence matrix of the complete graph on  $n$  vertices labelled  $1, \dots, n$ . Let each edge be weighted according to the absolute value of the difference of its end-vertex labels and arrange the columns of  $B_n$  in descending order of the weights of the corresponding columns. Thus, for  $K_4$ ,  $B_4$  would be specified as:

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 \\ -1 & 0 & -1 & 0 & 0 & -1 \end{pmatrix}$$

Specify a second matrix  $C_n$  of size  $\frac{1}{2}n(n - 1) \times n$  as follows: the first row is  $e_2$ , the  $n$ -unit vector which is zero except for a one in the second position; rows two and three are  $e_3$ , and so on, with rows  $\frac{1}{2}k(k - 1) + 1, \dots, \frac{1}{2}k(k + 1)$  being  $k$  copies of  $e_{k+1}$ , for  $k = 1, \dots, n - 1$ . Now let  $W_n$  be a diagonal matrix of indeterminants with the diagonal entry corresponding to edge  $ij$  in the ordering of the columns of  $B_n$  being  $x_{ij}$ , and consider the product,  $\Psi_n = B_n W_n C_n$ , which, for  $n = 4$  would be

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 \\ -1 & 0 & -1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_{14} & 0 & 0 & 0 & 0 & 0 \\ 0 & x_{13} & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{24} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & x_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & x_{34} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now  $n - 1 \times n - 1$  submatrices of  $C_n$  will be nonsingular if and only if they omit the first column and moreover correspond to subsets of edges of  $K_n$  having different edge weights. So applying the Cauchy-Binet theorem we have

**Theorem 5** *Let  $\Psi'_n$  denote  $B_n W_n C_n$  with its first row and column deleted. The multinomial in  $x_{ij}$  given by  $\det \Psi'_n$  enumerates all gracefully labelled trees on  $n$  vertices.*

To continue the  $K_4$  example, the determinant  $\det \Psi'_4$  will yield

$$-x_{1,4}x_{2,4}x_{2,3} + x_{1,4}x_{2,4}x_{3,4} - x_{1,4}x_{1,3}x_{1,2} + x_{1,4}x_{1,3}x_{2,3},$$

which is seen to enumerate the four graceful trees on four vertices and, simultaneously, the four placements of 3 rooks on the  $3 \times 3$  Klein bottle.

The matrix  $\Psi_n$  has a rather 'regular' structure which we will now exploit to derive a permanent to count  $n - 1$ -rook placements on the Klein bottle. This structure is illustrated for  $n = 4$ :



$$\Psi_4 = \begin{pmatrix} 0 & x_{1,4} & x_{1,3} & x_{1,2} \\ 0 & 0 & x_{2,4} & -x_{1,2} + x_{2,3} \\ 0 & 0 & -x_{1,3} & -x_{2,3} + x_{3,4} \\ 0 & -x_{1,4} & -x_{2,4} & -x_{3,4} \end{pmatrix}.$$

In general, we may define two  $n \times n$  matrices,  $Y_n = [y_{ij}]$  and  $Z_n = [z_{ij}]$ , by

$$y_{ij} = \begin{cases} x_{i,n-j+i+1} & \text{if } i < j \\ 0 & \text{if } i \geq j \end{cases} \quad z_{ij} = \begin{cases} x_{j-n+i-1,i} & \text{if } i > n - j + 1 \\ 0 & \text{if } i \leq n - j + 1 \end{cases}$$

and then  $\Psi_n = Y_n - Z_n$ .

Now the multinomial enumeration of graceful trees can be arranged as an alternating sum, so that setting all the indeterminants  $x_i$  to one would cause the determinant to disappear. This appears to prevent the use of  $\Psi_n$  as a counting function for graceful trees. However, if we take instead the sum  $Y_n + Z_n$  and take the permanent then the oppositely oriented cycles in spanning subgraphs of  $K_n$ , instead of cancelling out as in the Matrix Tree Theorem, are both included in the enumeration (a description of the cancellation which takes place is given by Zeilberger [5] .)

**Theorem 6** *If  $R_n$  denotes the matrix  $Y_n + Z_n$  with first row and column deleted then per  $R_n$  enumerates the ways to place  $n - 1$  rooks on the  $n - 1 \times n - 1$  Klein bottle. If all the indeterminants  $x_{ij}$  are set to one then the permanent gives the leading term of the rook polynomial for the Klein bottle.*

As a final example, we give the enumeration multinomial for  $n = 5$ :

$$\begin{aligned} x_{1,5}x_{2,5}x_{3,5}x_{4,5} & - x_{1,5}x_{2,5}x_{3,5}x_{3,4} - x_{1,5}x_{2,5}x_{1,3}x_{4,5} + x_{1,5}x_{2,5}x_{1,3}x_{3,4} - x_{1,5}x_{2,5}x_{2,4}x_{2,3} \\ & + x_{1,5}x_{2,5}x_{2,4}x_{3,4} + x_{1,5}x_{1,4}x_{2,4}x_{2,3} - x_{1,5}x_{1,4}x_{2,4}x_{3,4} - x_{1,5}x_{1,4}x_{1,2}x_{3,5} \\ & + x_{1,5}x_{1,4}x_{1,3}x_{1,2} + x_{1,5}x_{1,4}x_{2,3}x_{3,5} - x_{1,5}x_{1,4}x_{1,3}x_{2,3}, \end{aligned}$$

which can be tested against fig. 3(b) to confirm that each term specifies a legal rook placement (remembering that edges are to be oriented towards vertex 5, so that  $x_{1,4}$ , say, may specify the 1,4 cell or the 4,1 cell of the Klein bottle.)

Setting the  $x_{ij}$  to one allows the leading term of the Klein bottle rook polynomial to be evaluated relatively easily as tabulated in table 1. We do not know of a way of evaluating the permanent involved in closed form. A larger question would seem to be:

**Question 2** *Determine the rook polynomials for the Klein bottle.*

n	# rook placements	# graceful trees
1	1	1
2	1	1
3	2	2
4	4	4
5	12	12
6	40	40
7	168	164
8	784	752
9	4272	4020
10	25504	23576
11	171168	155632
12	1242688	1112032
13	9915072	8733628
14	84714112	73547332
15	783454848	670789524

Table 1: number of ways of placing  $n$  rooks on an  $n \times n$  Klein bottle and the number of gracefully labelled trees on  $n$  vertices. The latter is A033472 in [4] where it is given upto  $n = 22$ .

A complete solution to the question should perhaps provide an analogue of Cheon et al's formula, given in theorem 2.

Meanwhile, the problem of determining the rook polynomials for the projective plane has received no attention to our knowledge and remains completely open.

As a final thought, the Laguerre equation

$$x \frac{d^2y}{dx^2} + (1-x) \frac{dy}{dx} + y = 0,$$

has solutions defined in terms of the Laguerre polynomials

$$\begin{aligned} L_0(x) &= 1 \\ L_1(x) &= -x + 1 \\ L_2(x) &= \frac{1}{2}(x^2 - 4x + 2) \\ L_3(x) &= \frac{1}{6}(-x^3 + 9x^2 - 18x + 6), \end{aligned}$$

and so on, the relationship with equation 1 being clear. It would be interesting to ask if the rook polynomials for the Klein bottle and projective plane are similarly related to solutions of some associated differential equations.

## Acknowledgement

This work has benefitted from some insightful comments of my colleague Jon Selig.

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