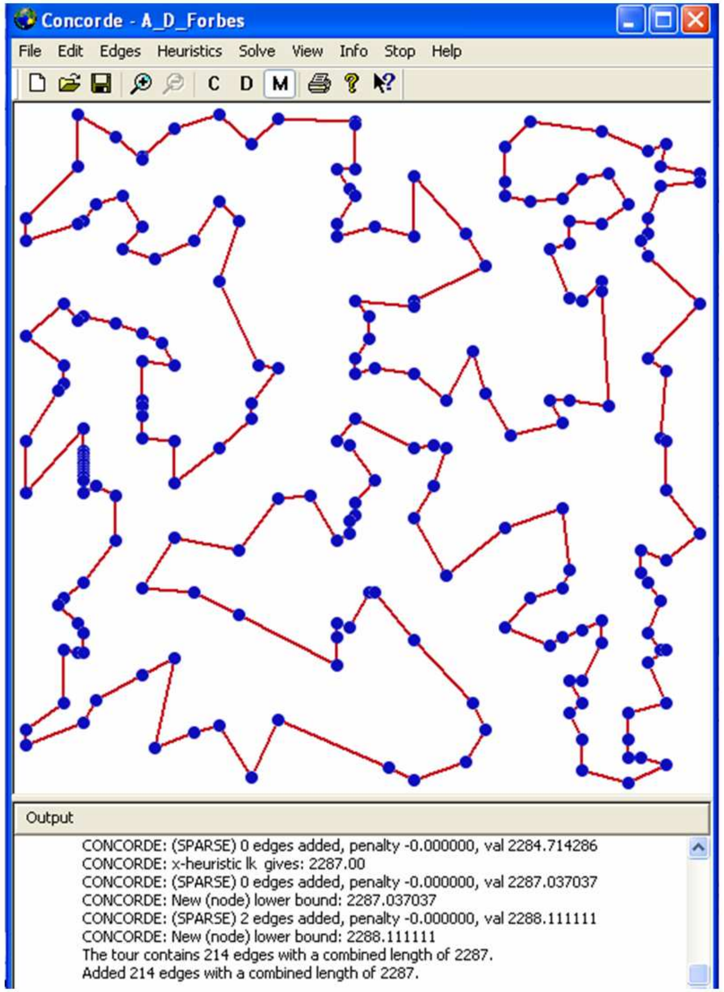
The Beardwood–Halton–Hammersley Theorem *Let* $\{X_1, \ldots, X_n\}$, $n \geq 1$, *be a set of random variables in* $\mathbb{R}^d$, *independently and identically distributed with bounded support. Then the length* L_n *of a shortest TSP tour through the points* X_i *satisfies*

$$L_n/n^{(d-1)/d} \to \beta_d \int_{\mathbb{R}^d} f(x)^{(d-1)/d} dx, \quad \textit{with probability 1, as } n \to \infty,$$

where $f(x)$ *is the absolutely continuous part of the distribution of the* X_i *and* β_d *is a constant which depends on d but not on this distribution.*



The easiest (2-dimensional) case of the theorem has the X_i uniformly distributed in the unit square: they have bounded support (the distribution is zero outside a bounded region), the integrand becomes $1^{1/2} = 1$ and the integral, being the volume of a unit cube, evaluates to 1. So the theorem says that a uniform distribution of n points in the unit square will, 'almost surely', be joined in a shortest Travelling Salesman Problem [TSP] circuit, or 'tour', whose length is asymptotic to $\beta_2\sqrt{n}$, as $n \to \infty$. More generally, to any continuous random variable X there corresponds, by definition, a *probability density function*, f, whose integral over an interval is the probability that X takes a value in that interval. In the theorem, the 'absolutely continuous part' of the distribution of the X_i can be thought of, roughly, as a version of f which excludes 'spikes' in the distribution of the X_i, i.e., measure zero subsets of $\mathbb{R}^n$ which occur with non-zero probability.

The illustration on the right is based on the well-known logarithmic distribution of prime numbers: the probability that an integer n is prime is asymptotic to $1/\log n$. The grid displays 210^2 consecutive integers arranged in rows of 210; of these a total of 214 are prime numbers and are marked as vertices (the diagram depicts the successful hunt for ten consecutive primes in arithmetic progression, as cited in Theorem no. 32). The TSP tour shown here has been calculated as the optimal one using the Concorde program (www.math.uwaterloo.ca/tsp/concorde.html). It has length 2287 units. Although these vertices certainly do not come from sampling a uniform distribution, we can still get some kind of an estimate for β_2 as though they were: $2287/(\sqrt{214} \times 210) \approx 0.74$. The best current estimates, based on simulations, range from 0.7119 to 0.7124.

Jillian Beardwood and John H. Halton were D.Phil. students of John M. Hammersley (1920–2004) at Oxford during the 1950s.

Web link: www.nap.edu/read/2026 (Chapter 8, by J. Michael. Steele)

Further reading: *The Traveling Salesman Problem: a Computational Study* by D.L. Applegate, R.E. Bixby, V. Chvátal and W.J. Cook, Princeton University Press, 2006. Chapter 1 can be previewed here: press.princeton.edu/chapters/s8451.pdf (2.44MB).