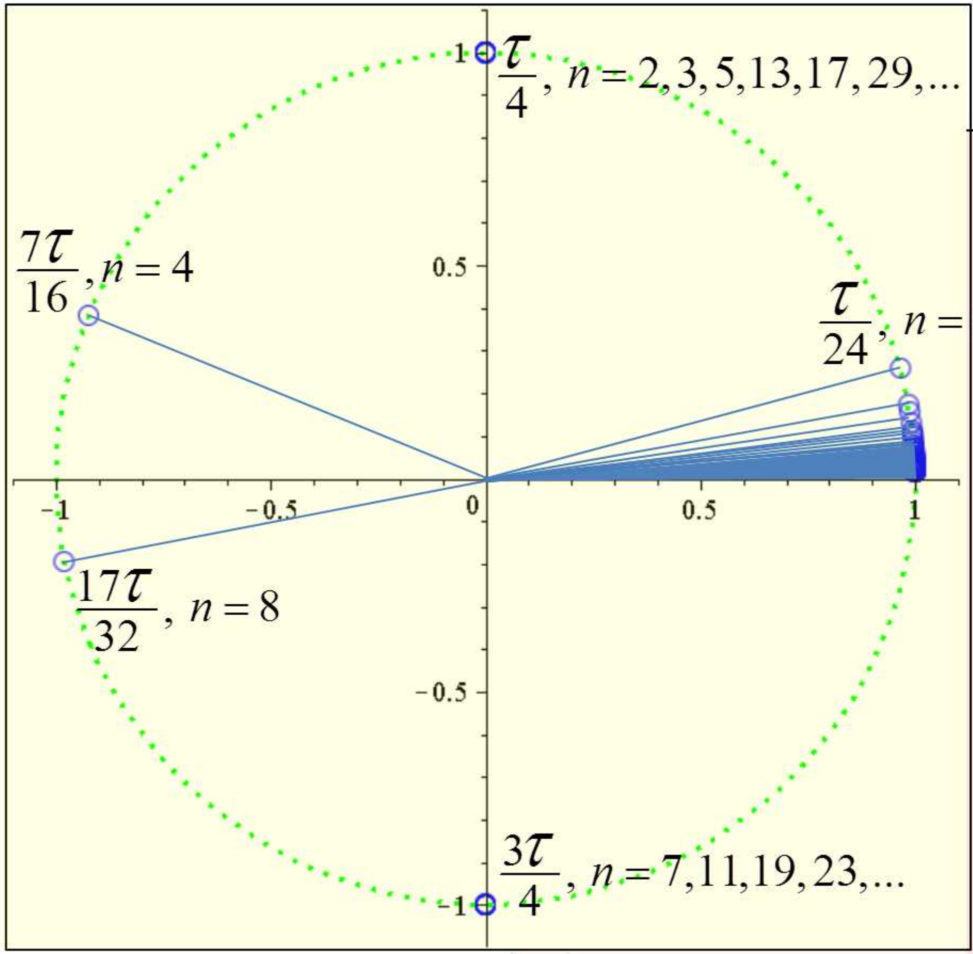
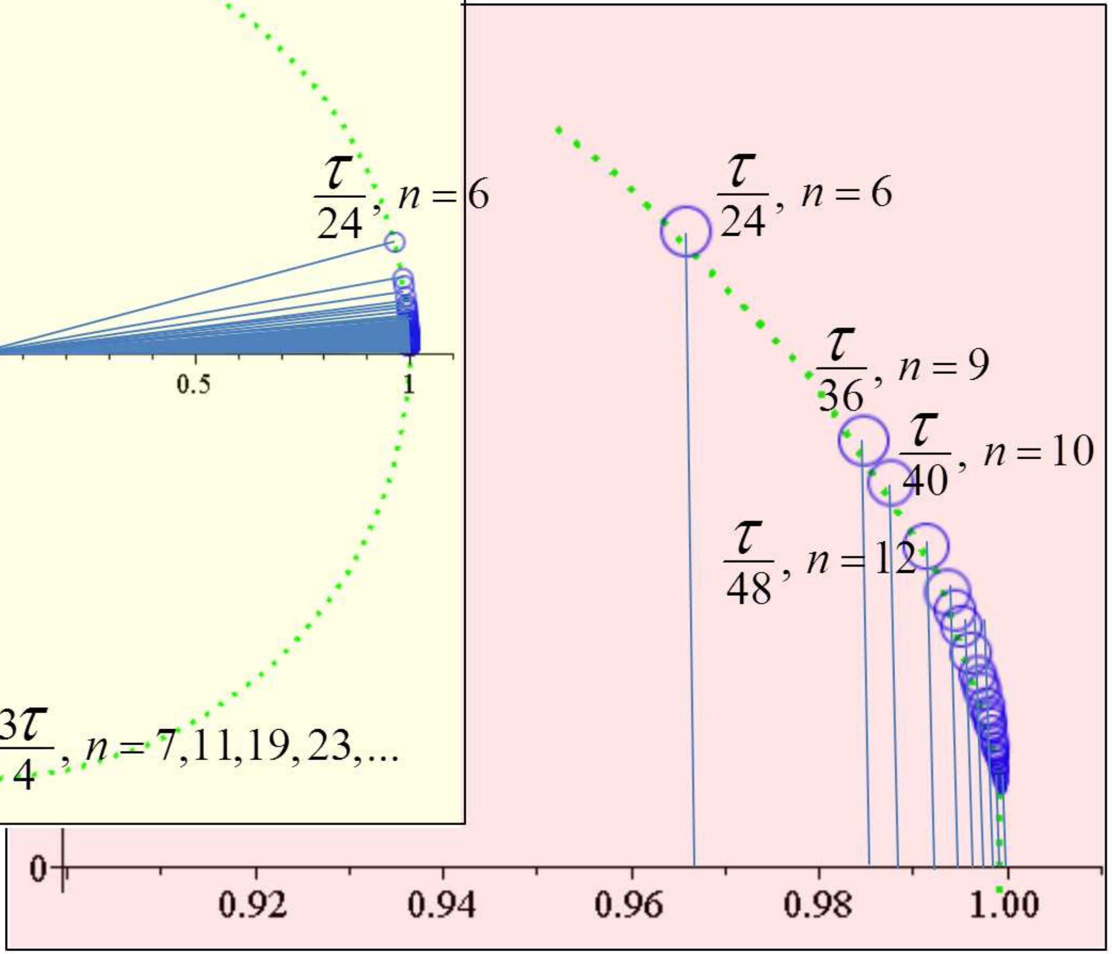
THEOREM OF THE DAY

Willans' Formula *The number of primes not exceeding a positive integer n may be calculated as*

$$\pi(n) = \left\lfloor \frac{1}{2}\left(n - 1 + \sum_{k=1}^{n-1} \cos\left(\frac{(k! - k)\tau}{2(k+1)}\right)\right)\right\rfloor .$$





We have slightly adapted the original statement of this formula:

$$\pi(n) = -1 + \sum \left\lfloor \cos^2\left(\frac{1 + (k-1)!}{2k}\tau\right)\right\rfloor ,$$

($\lfloor x \rfloor$ denotes the greatest integer not exceeding x) which is a direct corollary of Wilson's Theorem: $(1 + (k-1)!)\tau/2k$ is a multiple of $\tau/2$ if and only if k is prime, so the $\lfloor \cos^2 \rfloor$ will be 1 for primes and zero for composite numbers. Replacing cosine with sine and $\tau/2$ with $\tau/4$ produces the same result. But while cosines of non-multiples of $\tau/2$ approach 1, the sines of non-multiples of $\tau/4$ approach zero (see the plots on the left). For small angles x, $\sin x \approx x$, so summing over composite k contributes

$$\begin{aligned}\sum \sin^2\left(\frac{\tau}{4k}\right) &\approx \frac{\tau^2}{16}\sum \frac{1}{k^2}\\ &< \frac{\tau^2}{16}\left(\frac{\tau^2}{24} - 1 - \frac{1}{4} - \frac{1}{9}\right)\\ &< 1,\end{aligned}$$

invoking the Basel Problem and excluding the first few non-composites. This means we can move the $\lfloor \ldots \rfloor$ outside the $\sum$, and finally we swap $\sin^2$ with cos for our statement of Willans' formula by using the appropriate double-angle formula.

This result was published in 1964 by C.P. Willans, who was quick to point out that it did not contribute to our understanding of the distribution of the primes (being merely a repackaging of Wilson's Theorem). However, it remains an elegant and much-quoted means of accessing small values of $\pi(n)$ without explicit reference to primality testing (as in, say, the Sieve of Eratosthenes).

Web link: recursed.blogspot.co.uk/2013/01/no-formula-for-prime-numbers.html. Click here for an example of Willans' formula in action.

Further reading: *The Little Book of Bigger Primes* by Paulo Ribenboim, 2nd edition, Springer-Verlag, 2004, Chapter 3.