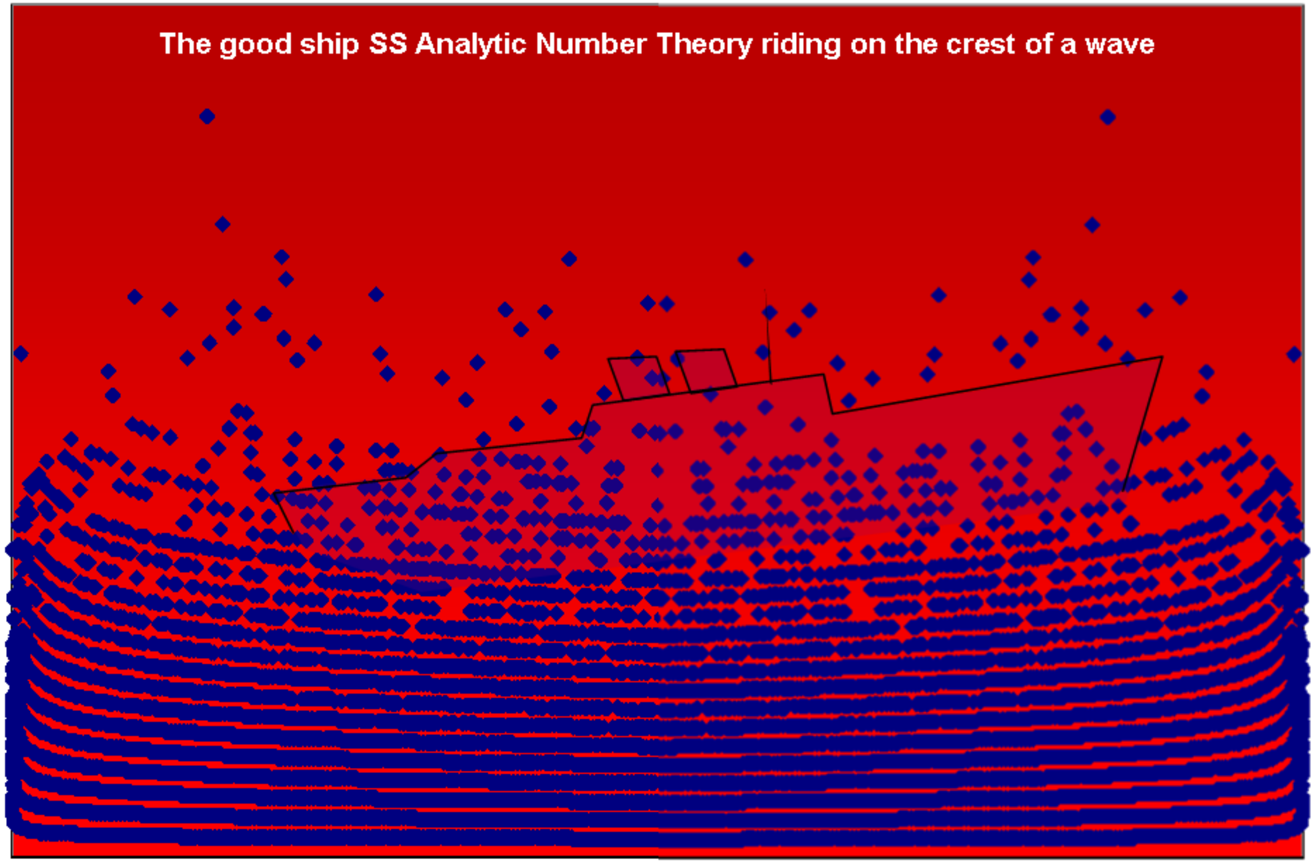
The Small Prime Gaps Theorem *For a prime number p denote by p_{next} the least prime greater than p. Then*

$$\liminf_{p\to\infty} \frac{p_{\text{next}} - p}{\log p} = 0.$$



Update: November 2013
It is now known that

$$\liminf_{p\to\infty} p_{\text{next}} - p < \text{constant}.$$

Zhang, then Maynard, build on the Goldston–Pintz–Yıldırım approach for far stronger conclusions. For details and further updates see Polymath8 at polymathprojects.org/.

In the image above, the left-hand half of the sea is a plot of the values of $(p_{\text{next}} - p)/\log p$ for the first 10000 primes (the plot is reversed to complete the right-hand side). The points in the lowest blue curve descend very gradually (as if towards the sea bed): these are the twin primes, those with $p_{\text{next}} - p = 2$. Does this curve extend forever (the 'twin primes conjecture')? Today's theorem says that at any rate the sea, as a whole, approaches arbitrarily close to the sea bed.

This 2005 breakthrough of Dan Goldston, János Pintz and Cem Yıldırım*, shows that prime gaps of length $\varepsilon \times \log p$ (i.e. ε of the average) occur infinitely often, *for arbitrarily small* ε (previous best: 1988, Helmut Maier's bound of $\varepsilon \approx 1/4$).

*the Turkish 'i' is pronounced like the 'e' in 'open'.

Web link: www.ams.org/bull/2007-44-01/: click on the excellent article by Kannan Soundararajan.

Further reading: *Closing the Gap: the quest to understand prime numbers* by Vicky Neale, Oxford University Press, 2017.