



THEOREM OF THE DAY

The Large Prime Gaps Theorem (a Theorem under Construction!) Let p_n denote the n -th prime number. Then for sufficiently large X ,

$$\max_{p_{n+1} \leq X} (p_{n+1} - p_n) \gg \frac{\log X \log \log X \log \log \log \log X}{\log \log \log X}.$$

A THEOREM UNDER CONSTRUCTION



prime	residue	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	
2	1	Red		Red		Red		Red		Red		Red		Red		Red		Red		Red		Red		Red		Red		Red		Red		
3	1	Yellow			Yellow			Yellow			Yellow			Yellow			Yellow			Yellow			Yellow			Yellow			Yellow			
5	3			Yellow					Yellow				Yellow					Yellow					Yellow					Yellow				
7	6						Green						Green					Green									Green					
11	2		Blue										Blue											Blue								
13	1	Purple												Purple													Purple					
17	12											Purple																		Purple		
included?		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✗

It is easy to find arbitrarily long sequences of consecutive composite numbers: $n! + 2, n! + 3, \dots, n! + n$, for example. But our theorem concerns $\max_{p_{n+1} \leq X} (p_{n+1} - p_n)$, which we denote by $G(X)$, requiring our sequence to be located below a given X . An elegant ‘sieving’ device makes a start: define $Y(x)$ to be the largest integer y for which one may select residue classes $a_p \pmod p$, one for each prime $p \leq x$, whose union contains the whole set $\{1, \dots, y\}$. Denote by $P(x)$ the product of all primes not exceeding x . Then **Lemma:** $G(P(x) + Y(x) + x) \geq Y(x)$. Use of the Prime Number Theorem and a rough upper bound on Y turns this around to give $G(X) \geq Y((1 + o(1)) \log X)$, as $X \rightarrow \infty$, (with $o(1)$ being a contribution that becomes vanishingly small).

The table above illustrates the definition of $Y(x)$ for $x = 17$ and reveals why the Lemma is true. The residue classes chosen, $a_2 = 1 = a_3, a_5 = 3$, etc, include every positive integer below 26. In fact, $Y(17) = 25$ precisely, so this is the best we can do. The Lemma works as follows: use the Chinese Remainder Theorem to find a solution m , $17 < m \leq 17 + P(17)$, to the congruences $m \equiv -1 \pmod 2, m \equiv -1 \pmod 3, m \equiv -3 \pmod 5, \dots, m \equiv -12 \pmod{17}$. Consider $m + k$, for $1 \leq k \leq Y(17)$. By definition, $k \equiv a_p \pmod p$ for some $p \leq 17$, and some residue a_p . But then $m + k \equiv -a_p + a_p = 0 \pmod p$. So p divides $m + k$; and $p \neq m + k$ because $p \leq x < m < m + k$. So all of $m + 1, m + 2, \dots, m + Y(17)$ are composite.

Denote by \log_n the n -th iterated (natural) log. The above bound, $G(X) \gg \log X \log_2 X \log_4 X / \log_3 X$, is compared on the right with previous bounds (the ‘ \gg ’ means, roughly, ‘up to a constant multiple’; we have used constant = 1). They are dwarfed by Harald Cramér’s conjectured $\limsup G(X) / (\log X)^2 = 1$ which is plotted here as a \log_2 to keep it in the picture!

This 2015 advance is due to Kevin Ford, Ben Green, Sergei Konyagin, James Maynard and Terence Tao.

Web link: arxiv.org/abs/1412.5029 (the above sieving lemma appears as Lemma 1.1.)

Further reading: *The Little Book of Bigger Primes* by Paulo Ribenboim, 2nd edition, Springer-Verlag, 2004, Chapter 4.

