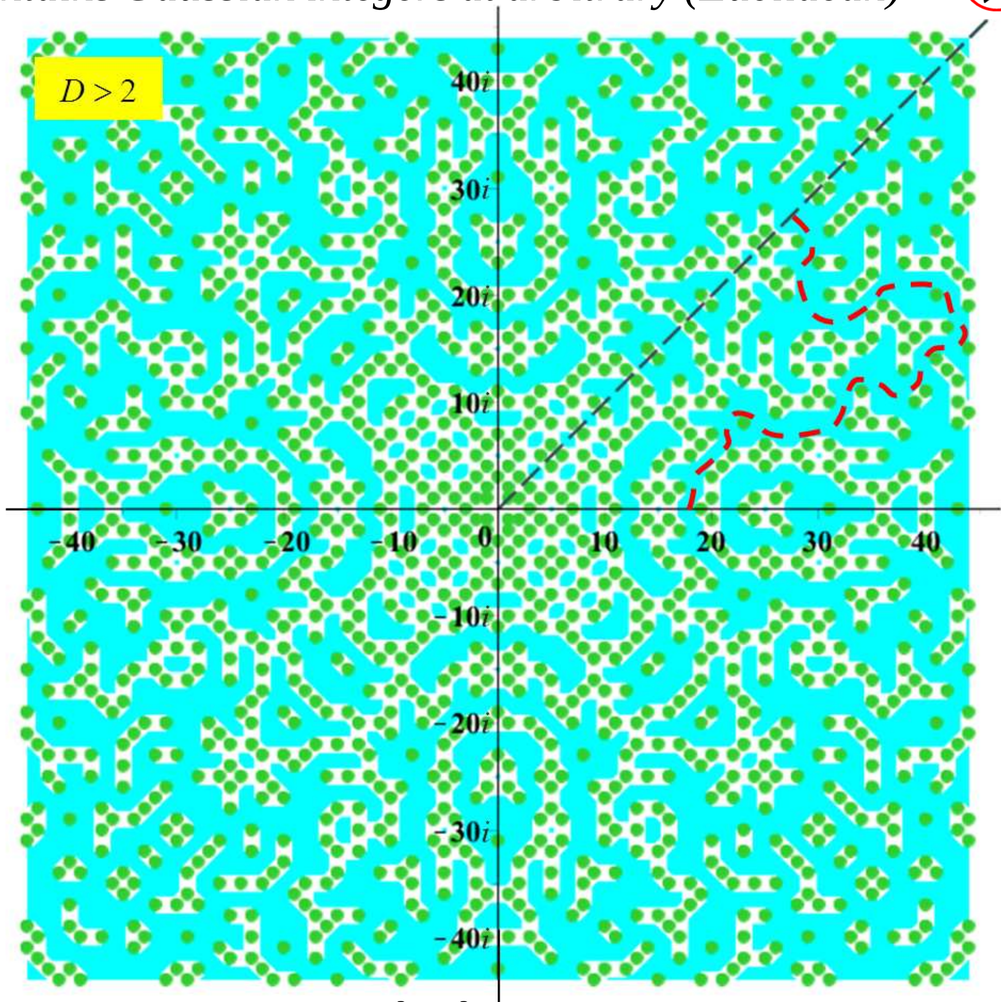
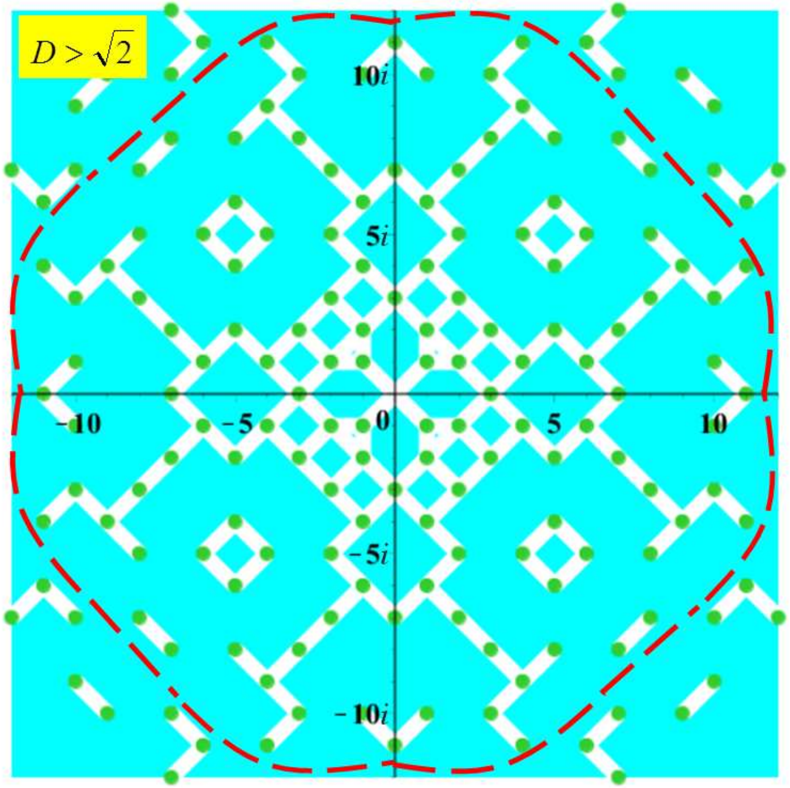
THEOREM OF THE DAY

A Theorem about Gaussian Moats (a Theorem under Construction!) *Any straight line in the complex plane passing through two distinct Gaussian integers contains Gaussian integers at arbitrary (Euclidean) distance from any Gaussian prime.*



A Gaussian integer is a complex number of the form $z = a + ib$ with a and b integers; it is prime if it cannot be factored. For example, $5 = 5 + 0i$ is a Gaussian integer, but it is not a Gaussian prime because it factors as $5 = (1 + 2i)(1 - 2i) = (2 + i)(2 - i)$.

A THEOREM UNDER CONSTRUCTION



Observe that these 2 factorisations differ only up to multiplying by $\pm\{1, i\}$: like ordinary integer factorisation, Gaussian integer factorisation is unique. In fact this reveals which Gaussian integers are prime: let $a + ib$ be prime and suppose $a^2 + b^2$ factorises into ordinary integers $a^2 + b^2 = p \times n$, with p an ordinary prime. Now $a^2 + b^2$ also factorises as $(a+ib)(a-ib)$, so one of $a + ib$ and $a - ib$ must be equal to p or pi, so that one of a and b is zero. We conclude that Gaussian primes $z = a + ib$, are those for which $a, b \neq 0$ and $a^2 + b^2$ a is prime number, and those which are $\pm\{1, i\}$ times an ordinary prime, which moreover cannot be congruent to 1 mod 4, since otherwise Fermat's 2-Squares theorem factorises z as $x^2 + y^2 = (x + iy)(x - iy)$. Now suppose we plot the Gaussian primes, together with the origin, in the complex plane. Join any two plotted points with an edge if they are at most, say, D in Euclidean distance from each other. Can we walk along consecutive edges from the origin to infinity? Or is there some 'moat' around the origin whose 'width' exceeds D? (Note that there is 8-fold symmetry in our Gaussian primes plot, so we need only find a moat through a 45° sector.)

Still unsolved, Basil Gordon's 1962 question "does some finite value of D allow walks to infinity?" has a negative answer for straight-line walks, as shown in 1998 by Ellen Gethner, Stan Wagon and Brian Wick, who also showed $D \geq \sqrt{26}$ in general.

Web link: www.maa.org/programs/maa-awards/writing-awards/a-stroll-through-the-gaussian-primes

Further reading: *The Edge of the Universe: Celebrating Ten Years of Math Horizons* by Deanna Haunsperger and Stephen Kennedy (eds.), Mathematical Association of America, 2006.

Created by Robin Whitty for www.theoremoftheday.org