THEOREM OF THE DAY

The Fifteen Theorem *If a positive-definite quadratic form defined by a symmetric, integral matrix takes each of the values 1, 2, 3, 5, 6, 7, 10, 14, 15, then it takes all positive integer values.*

	w	x	y	z
w	1			
x		1		
y			1	
z				1

$w^2 + x^2 + y^2 + z^2$

w	x	y	z	
0	0	0	1	**1**
0	0	1	1	**2**
0	1	1	1	**3**
0	0	1	2	**5**
0	1	1	2	**6**
1	1	1	2	**7**
1	1	2	2	**10**
0	1	2	3	**14**
1	1	2	3	**15**

	w	x	y	z
w	1	0	1	2
x	0	2	0	-2
y	1	0	2	0
z	2	-2	0	11

$w^2 + 2x^2 + 2y^2 + 11z^2$

$+4wz$ $+2wy$ $-4xz$

w	x	y	z	
1	0	0	0	**1**
0	0	1	0	**2**
1	1	0	0	**3**
1	0	1	0	**5**
1	-1	0	-1	**6**
1	1	1	0	**7**
0	1	2	0	**10**
1	1	0	1	**14**
1	1	2	0	**15**

A *quadratic form* is a sum of products of variables, with each product comprising precisely two variables (possibly the same, in which case the product is a square). It is *positive-definite* if the result is a non-negative number, no matter what values the variables take. In many cases such a quadratic form may be defined by a square, symmetric matrix, as shown here, and the theorem applies when this matrix has only integer entries (an *integral matrix*).

Two 4-variable examples are shown here. On the left, the theorem confirms that $w^2 + x^2 + y^2 + z^2$ is *universal* (takes all positive integer values) which is Lagrange's celebrated 'four squares theorem' of 1770. On the right a more elaborate matrix is again confirmed to be universal; for example 1729 (famously the smallest positive integer which can be expressed as a sum of two cubes in two different ways) is given when w, x, y and z take the values, 7,-6,-6 and 10, respectively.

The Fifteen Theorem was the astonishing discovery of John Conway and William Schneeberger in 1993. Their complex proof was never published, being superseded in 2000 by the work of a PhD student of Andrew Wiles, Manjul Bhargava. He brilliantly simplified the proof and developed other universality results: for example, a list of 29 integers which guarantees universality even for those positive-definite quadratic forms, e.g., $x^2 + xy + y^2$, which are *not* defined by a symmetric integral matrix.

Web link: www.fen.bilkent.edu.tr/~franz/mat/15.pdf

Further reading: *The Sensual (Quadratic) Form* by J.H. Conway with F.Y.C. Yung, The Mathematical Association of America, 1998.