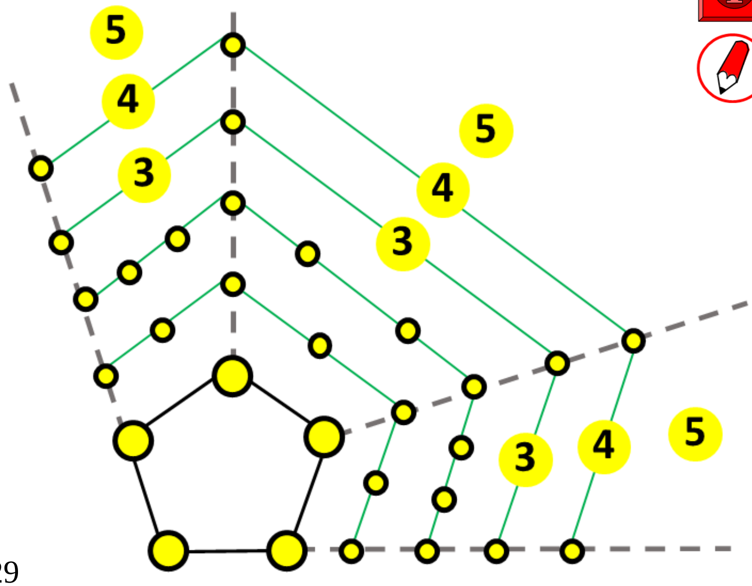
THEOREM OF THE DAY

The Polygonal Number Theorem *For any integer $m > 1$, every non-negative integer n is a sum of $m+2$ polygonal numbers of order $m+2$.*



For a positive integer m, the polygonal numbers of order $m + 2$ are the values

$$P_m(k) = \frac{m}{2}\left(k^2 - k\right) + k, k \geq 0.$$

The first case, $m = 1$, gives the **triangular numbers**, $0, 1, 3, 6, 10, \ldots$. A general diagrammatic construction is illustrated on the right for the case $m = 3$, the **pentagonal numbers**: a regular $(m + 2)$−gon is extended by adding vertices along 'rays' of new vertices from $(m + 1)$ vertices with $1, 2, 3, \ldots$ additional vertices inserted between each ray.

How can we find a representation of a given n in terms of polygonal numbers of a given order $m+2$? How do we discover, say, that $n = 375$ is the sum $247+70+35+22+1$ of five pentagonal numbers? What follows a piece of pure sorcery from the celebrated number theorist Melvyn B. Nathanson!

1. Assume that $m \geq 3$. Choose an odd positive integer b such that — $b = 29$

   (1) We can write $n \equiv b + r \pmod m, 0 \leq r \leq m - 2$; and — $n = 375 \equiv 29 + 1 \pmod 3$

   (2) If $a = 2\left(\frac{n - b - r}{m}\right) + b$, an odd positive integer by virtue of (1), then — $a = 259$

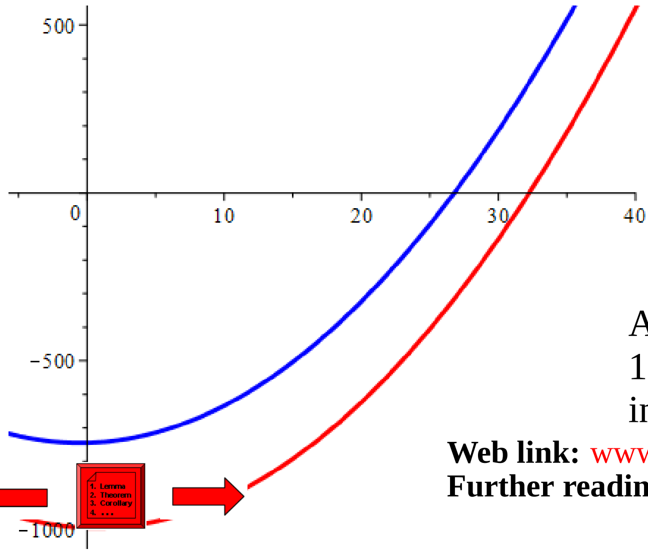
   $$b^2 - 4a < 0 \quad \text{and} \quad 0 < b^2 + 2b - 3a + 4. \qquad (*)$$ — $841 - 1036 < 0,\ 0 < 841 + 58 - 777 + 4$

2. Invoke **Cauchy's Lemma:** *If a and b are odd positive integers satisfying $(*)$ then there exist nonnegative integers s, t, u, v such that*

   $$a = s^2 + t^2 + u^2 + v^2 \quad \text{and} \quad b = s + t + u + v.$$ — $259 = 13^2 + 7^2 + 5^2 + 4^2$ and $29 = 13 + 7 + 5 + 4$

3. From the definition of a in step 1(2), write $n = \frac{m}{2}(a - b) + b + r$

   $$= \frac{m}{2}(s^2 - s) + s + \ldots + \frac{m}{2}(v^2 - v) + v + r.$$ — $375 = 247 + 70 + 35 + 22 + 1$



How can we be sure (1) and (2) in step 1 are possible? We appeal to the quadratic formula, applied to the two quadratics in $(*)$ (plotted for our example on the left). The roots specify an interval $[b_1, b_2]$ from which to select the value of b. If $b_2 - b_1 \geq 4$, then the interval must contain consecutive odd integers: together they will supply enough modulo values for the equation in 1(1) to be satisfied. Now $b_2 - b_1 \geq 4$ is guaranteed for large enough n, specifically $n \geq 120m$. Luckily for all smaller values of n the theorem is known from tabulations made in the 19th century. Step 1 also needs $m \geq 3$; this also is aleady established as explained below.

A typical piece of unproven genius from Pierre de Fermat in 1638. Lagrange proved $m = 2$ in 1770 (the Four Squares Theorem). Gauss proved $m = 1$ in 1796 (his **Eureka Theorem**). Finally in 1815 came Cauchy's proof of $m \geq 3$, dramatically shortened in 1987 by Nathanson!

Web link: www.fields.utoronto.ca/programs/scientific/11-12/Mtl-To-numbertheory/ (11.45 on Sunday October 9)

Further reading: *Additive Number Theory, The Classical Bases,* by Melvyn B Nathanson, Springer, 1996.