Hyperperfect numbers

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This is a little article which is mostly about the expression

$$(1+k)^{m-1}\left((1+k)^m - k\right). \tag{1}$$

If $k = 1$ the second factor is a Mersenne number and is known (as of July 2026) to be prime for 52 values of m. For all these values, expression (1) evaluates to a perfect number, that is, one which is equal to exactly half the sum of its divisors. For $k = 1$ and $m = 2$, for example, the value is 6 which is half of $1 + 2 + 3 + 6$. The reader will probably greet, as old friends, the first few perfect numbers:

$$6, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128, \ldots,$$

and will perhaps have an opinion about whether or not this sequence continues indefinitely.

For a positive integer n, the sum of divisors function, $\sigma(n)$, is defined to be the sum of the positive divisors of n. A perfect number is thus a value of n for which $\sigma(n) = 2n$. Then, for the equation

$$\sigma(n) = \frac{k+1}{k}n + \frac{k-1}{k}, \tag{2}$$

the solutions when $k = 1$ are the perfect numbers. The solutions for a general positive integer value of k are called the k-hyperperfect numbers. They were introduced in Daniel Minoli and Robert Bear, "Hyperperfect numbers", *Pi Mu Epsilon J.*, Vol. 6, No. 3, 1975, pp. 153-157. Equation (2) rearranges to give the alternative, more often used, definition: n is a k-hyperperfect number if and only if it satisfies the equation:

$$n = 1 + k\left(\sigma(n) - n - 1\right). \tag{3}$$

Calculations using equations (2) and (3) are often made easy by the basic properties of the σ function:

Lemma 1 For n a positive integer

1. if n is prime then $\sigma(n) = 1 + n$;
2. $\sigma(n)$ is multiplicative: if n is a product of two coprime numbers, say $n = ab$, then $\sigma(n) = \sigma(a).\sigma(b)$.

We can rewrite equation (3) to define the hyperperfect numbers in a rather more natural way, by using a version of the σ function that is less standard (and is no longer multiplicative, for example). Let $\sigma^*(n)$ denote the sum of the proper, nontrivial divisors, that is, $\sigma^*(n) = \sigma(n) - 1 - n$. The perfect numbers are then those n for which $n = 1 + \sigma^*(n)$, and n is a k-hyperperfect number if and only if

$$n = 1 + k\sigma^*(n). \tag{4}$$

We use this version of the definition to prove a simple but striking result of Minoli and Bear:

Lemma 2 If n has a divisor strictly between 1 and $1 + k$ then n cannot be k-hyperperfect.

Proof. Suppose that d divides n with $1 < d \leq k$. We observe that d cannot equal n, otherwise $n = d \leq k$ which is impossible since $n = 1 + k\sigma^*(n) > k$.

Suppose that d is a proper divisor of n. Then n/d is also a proper divisor of n. But then the definition of k-hyperperfect gives

$$n = 1 + k\sum_{d|n,1<d<n} d = 1 + k\left(\frac{n}{d} + \ldots\right) > n,$$

since $d \leq k$. So n cannot be k-hyperperfect. □

Lemma 2 is striking because it follows that no k-hyperfect number can be even when $k > 1$; whereas it is famously an open question whether there exist any 1-hyperperfect (that is, perfect) numbers that are *not* even. In other words, it is an open question whether or not the following celebrated injective mapping from Mersenne primes to perfect numbers extends to a surjection:

Theorem 3 (Euclid–Euler): An even positive integer n is perfect if and only if it can be written as

$$n = 2^{m-1}(2^m - 1), \tag{5}$$

for some m for which $2^m - 1$ is prime.

There are other ways to generalise perfect numbers. The great appeal of Minoli and Bear's definition lies in the following, as they put it, "astonishing relation" to the Euclid–Euler theorem:

Theorem 4 (Minoli and Bear, Nash): For $k \geq 1$, let $n = (1+k)^{m-1}((1+k)^m - k)$, where $(1+k)^m - k$ is prime. Then n is a k-hyperperfect number if and only if $1 + k$ is prime.

Minoli and Bear proved this result only for $k = 2$ but the proof for the general case follows in an almost identical fashion. It was written down in John C. M. Nash, "Hyperperfect numbers", *Pi Mu Epsilon J.*,Vol. 11, No. 5, 2001, p. 251 (by the way, this is Nash the son of John Forbes Nash). Neither state the 'only if' part but this follows at once from Lemma 2 since if $1+k$ is not prime then n in the theorem has a divisor strictly less than $1 + k$.

Theorem 4 for $k = 1$ and the Euclid-Euler theorem which, as far as we know, accounts for all 1-hyperperfect numbers, are tantalisingly closely connected. We might hope that theorem 4 would account for all k-hyperperfect numbers in every case where k is one less than a prime. This fails early on because $7^2.383.3203$ is 6-hyperperfect. In fact, for even values of k, things appear quite badly-behaved, some of the worse excesses being reported in Mariano Garcia, "Hyperperfect numbers with five and six different prime factors", *Fibonacci Quarterly*, Vol. 42, No. 4, 2004, pp. 292–294. Several of Garcia's even k values are one less than a prime.

The case $k = 2$ remained open for fifty years during which time the only known 2-hyperperfect numbers were those specified by theorem 4. However, in June 2026, on the SeqFAN Google Group ("a mailing list for Integer SEQuence FANs"), user Alex Violette announced the discovery of a 2-hyperperfect number that did not involve expression (1):

$$3^{16} \cdot 129140227 \cdot 256572153302107 \cdot 22520539685553710729893386637 3.$$

The three factors following the 3^{16} are primes. This makes it easy to confirm, via Lemma 1, that Alex Violette's find satisfies equation (2) with $k = 2$. As for the case $k = 4$, only values supplied by theorem 4 are known, as of July 2026, to be 4-hyperperfect but, mindful of the fate of $k = 2$, I do not think one would conjecture that this is always true.

The case of odd $k > 1$ is of course excluded from theorem 4. In this case we have a construction from Judson S. McCranie, "A study of hyperperfect numbers", *Journal of Integer Sequences*, Vol. 3, 2000, Article 00.1.3.

Theorem 5 (McCranie): For odd $k > 1$, let $p = (3k+1)/2$ and $q = 3k + 4$ be prime. Then $p^2 q$ is k-hyperperfect.

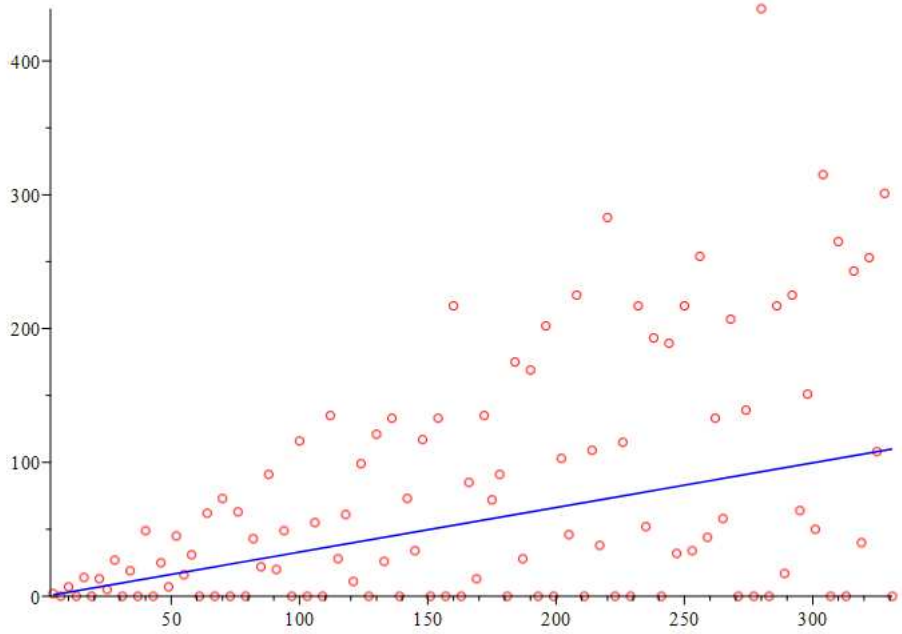
McCranie conjectured that no other k-hyperperfect numbers exist for odd $k > 1$ and this conjecture still stands as of July 2026. Moreover, there are many odd k for which the hypothesis of theorem 5 cannot hold ($k = 5$, for example). So, conjecturally, there are values of k for which no k-hyperperfect numbers exist. In particular, no 5-hyperperfect numbers are known.

We can explore particular values of k numerically. From equation (4), for n to be k-hyperperfect, we have $n = kt + 1$ for some $t \geq 1$. Then, again by equation (4), $kt + 1 = 1 + k\sigma^*(kt + 1)$, so

$$\sigma^*(kt + 1) = t. \tag{6}$$

The following plot explores the value $k = 3$. The straight line is $y = (x - 1)/3$. The value of $\sigma^*(3n + 1), n = 0, \ldots, 110$, is plotted against this line as a series of points. There is one value where

the two plots coincide, indicating a 3-hyperperfect number. This is the point $(3t+1, t)$ for $t = 108$, verifying identity (6) and confirming the assertion of theorem 5 that, with $p = (3k+1)/2 = 5$ and $q = 3k+4 = 13$ both prime, the value $p^2q = 325$ is a 3-hyperperfect number. As of July 2026 this is the only 3-hyperperfect number known, in agreement with McCranie's conjecture.



By contrast the following plot shows data for $k = 5$. Despite some 'near misses' no points on the straight line $y = (x-1)/5$ are σ^* values. There are some seemingly collinear distributions of σ^* values which invite investigation: perhaps these correspond in some variant of hyperperfect numbers whose definition is easier to satisfy? Not that this means they would necessarily be easier to characterise. After all, there is a collinear distribution of σ^* values on our horizontal axis and this corresponds to prime values of $5t+1$.

