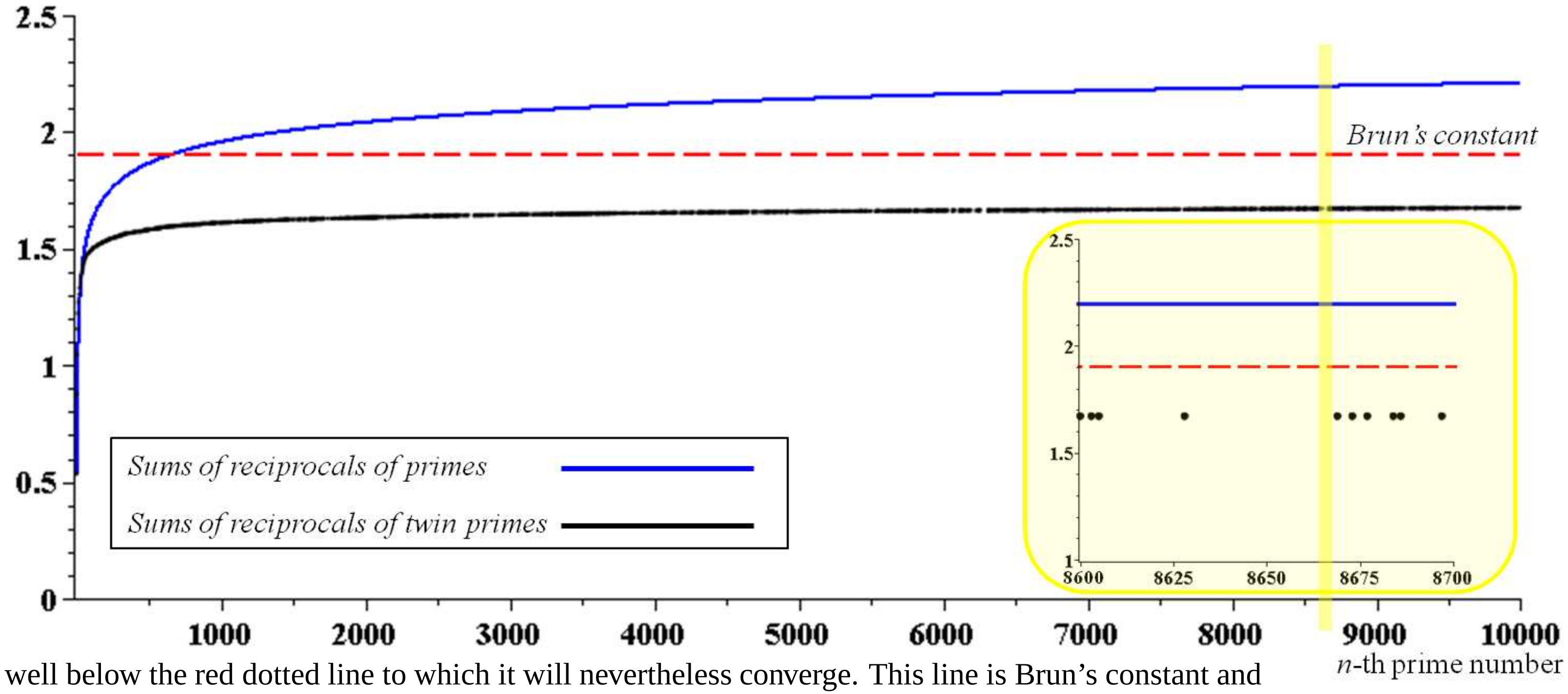
THEOREM OF THE DAY

Brun's Theorem *The series*

$$\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\frac{1}{19}+\frac{1}{29}+\frac{1}{31}+\frac{1}{41}+\frac{1}{43}+\frac{1}{59}+\frac{1}{61}+\ldots,$$

where the denominators are twin prime numbers, is convergent or finite.

Nomenclature varies: here, '**twin primes**' refers to those odd prime numbers separated from another odd prime by 2, the smallest possible amount, the pair being referred to as a **twin prime pair**. The first twin prime pairs are 3,5 and 5,7; the summation in Brun's Theorem avoids double counting the 5, but it is more common today to begin the sum $1/3 + 1/5 + 1/5 + 1/7 + 1/11 + \ldots$. This sum increases as shown in the graph on the right, in which the horizontal axis covers the first 10^4 primes. The lower, black curve records the increasing total of twin prime reciprocals, a new total being plotted each time a pair of twin prime reciprocals is added (the final pair included is 104681,104683).



The plot suggests that the curve will remain well below the red dotted line to which it will nevertheless converge. This line is Brun's constant and has been estimated, by Pascal Sebah using all twin primes less than 10^{16}, at 1.90216 05831 04. Although the black curve looks almost continuous there are only a little over 2500 twins among the first 10^4 primes. The magnified portion of the graph shown far right shows some large gaps between twins occurring between the 8600-th and 8700-th prime (there are no twin primes between 89071 and 89519, for example).

The statement of Viggo Brun's theorem given above is the actual title, translated from French, of his 1919 paper. The result was derived by placing a bound on the number of twin primes which could exist in larger and larger subsets of the positive integers. (Somewhat) more precisely (and using a simpler bound discovered a year later by Brun) let $\pi_2(n)$ denote the number of twin primes up to n and let p_n denote the n-th twin prime. Then $\pi_2(p_n) = n$, by definition, and Brun proved that, for large n and some constant c, $n = \pi_2(p_n) < cp_n/\log^2(p_n)$. Since $n < p_n$, this gives $n < cp_n/\log^2(n)$ which rearranges to $1/p_n < c/n\log^2(n)$, with the right-hand side giving the terms of a convergent series.

Brun's result is contrasted in the above plot with a classic result of Euler: *the sum of the reciprocals of the prime numbers diverges*. Although the upper, blue curve in the picture looks very similar in shape to the black curve there is no corresponding dotted line of convergence — the blue curve will eventually climb above the top of the page.

Web link: primes.utm.edu/glossary/page.php/TwinPrime.html. Pascal's Sabeh's work is here: numbers.computation.free.fr

Further reading: *Prime Numbers: The Most Mysterious Figures in Math* by David Wells, John Wiley & Sons, Inc., 1995