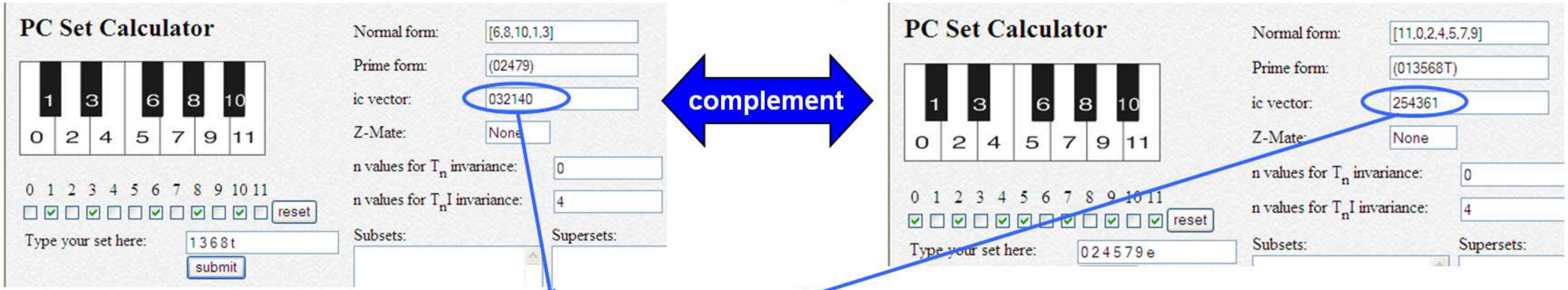
THEOREM OF THE DAY

The Generalised Hexachord Theorem *Suppose that S is a subset of the set of pitch classes comprising $\mathbb{Z}_n$, n even, with $|S| = s$; then the interval class vectors of S and its complement $\mathbb{Z}_n \setminus S$ differ componentwise by $|n - 2s|$, except for the last component, for which the difference is $|n/2 - s|$.*



$$|032140 - 254361| = 222221$$

The *interval class vector* of S is the vector of length $n/2$ whose i-th component is the number of pairs $a, b \in S$, $a < b$, for which $\min(b - a, n - b + a) = i$. For $\mathbb{Z}_{12}$, the integers modulo 12, this is illustrated above, using the opening bars of Chopin's op. 10, no. 5, for $i = 5$ (the upper, red, curves in bars 2–3) and $i = 2$ (the lower, green, curves). The word 'class' is used to indicate that no distinction is made between the interval from, say, B♭ *down* to E♭ (the first red curve) or from B♭ *up* to E♭; both intervals have value $5 = \min(10 - 3, 12 - 10 + 3)$.

The complete interval class vectors have been calculated here using David Walters' PC Set Calculator, which is attached to Gary Tucker's excellent Introduction to Pitch Class Analysis. The right-hand part of Chopin's étude uses only the five black-key notes of the piano, the complement of this set being all the white keys. The Generalised Hexachord Theorem then tells us that components 1–5 of the corresponding interval classes differ uniformly by $12 - 2 \times 5 = 2$.

The original Hexachord Theorem, due to the composer Milton Babbitt and theorist David Lewin in the 1950s, applies to subsets of exactly *six* notes out of the twelve note scale, the interval class vectors of this 'hexachord' or 'hexad' and its complement then being identical. A short proof using Fourier analysis was published by Emmanuel Amiot in 2006.

Web link: canonsrythmiques.free.fr/Quadrature/q05027.pdf (in French). For web-enabled Chopin research visit the superb Online Chopin Variorum Edition (OCVE): chopinonline.ac.uk/ocve/.

Further reading: *Music: a Mathematical Offering* by Dave Benson, Cambridge University Press, 2006 (chapter 9). Online version: visit homepages.abdn.ac.uk/mth192/pages/.