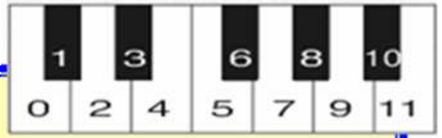
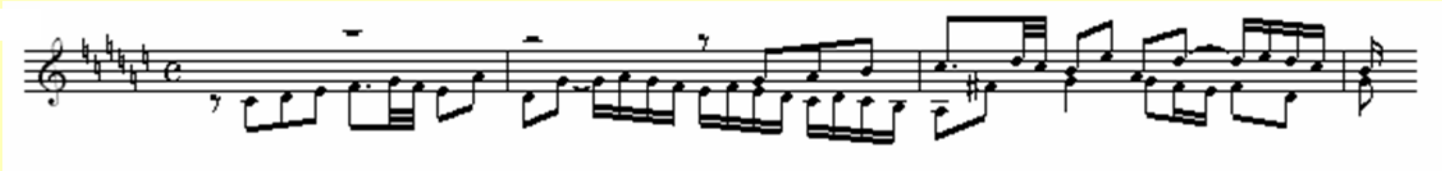
THEOREM OF THE DAY

Babbitt's Theorem *Suppose that S is a subset of the set of pitch classes comprising $\mathbb{Z}_n$, with interval class vector v. Let S' be the transposition of S by an interval $k \leq n/2$, that is, $S' = S + k$, with addition being modulo n. Then the number of pitches common to both S and S' is equal to the k-th entry of v: $|S \cap S'| = v_k$.*



J.S. Bach, Fugue in C from book 1 of the '48'

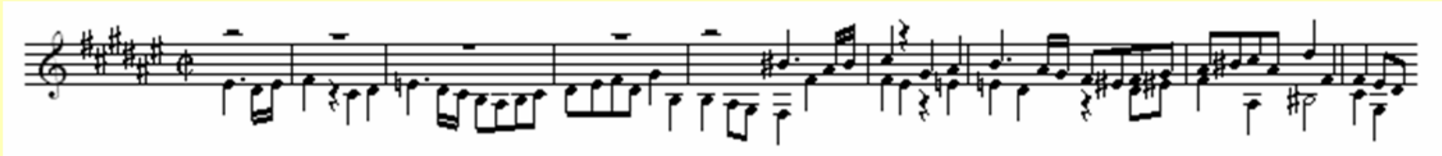


Subsets of $\mathbb{Z}_{12}$: subject: {0,2,4,5,7,9}
answer: {7,9,11,0,2,4}

Interval class vector:

1	2	3	4	5	6
1	4	3	2	5	0

J.S. Bach, Fugue in F# from book 2 of the '48'



Subsets of $\mathbb{Z}_{12}$: subject: {5,3,6,1,4,11,10,8}
answer: {0,10,1,8,11,6,5,3}

Interval class vector:

1	2	3	4	5	6
4	6	5	4	7	2

A little amateur musical analysis is applied here to two of the *real* fugues from Bach's Well-tempered Clavier (the '48'), that is, the fugues in which the subject is answered by an exact transposition by 7 pitch values (semitones) up, pitch class-equivalent to $12 - 7 = 5$ down. The interval class vectors indicate how many different pairs of pitch values are found at each distance apart, from 1 to 6 (7 and above being mapped to 5 and below). Of the major-key fugues in the 48, the majority have subjects using 6 of the 7 notes in the major scale. If the missing note value is 5 or 11 then the interval class vector is (1 4 3 2 5 0), as found here for the C major fugue (the one that follows *that* prelude); Babbitt's Theorem then guarantees maximum possible overlap between the notes of the subject and its answer: $v_5 = 5$. In the $F\sharp$ fugue from Book 2, some chromaticism allows an 8-pitch subject; again Bach's (deliberate?) choice of pitches means that the answer has maximum overlap: $v_5 = 7$.

This simple but effective result is attributed by music theorists to the American composer and analyst Milton Babbitt who would presumably have first discovered and used it in the 1950s.

Web link: www.mta.ca/pc-set/pc-set_new/

Further reading: *Music and Mathematics* by John Fauvel, Raymond Flood and Robin Wilson, Oxford University Press, 2003 (chapter 9).