The Sylvester-Gallai theorem: proofs from the BOOK

December 2022

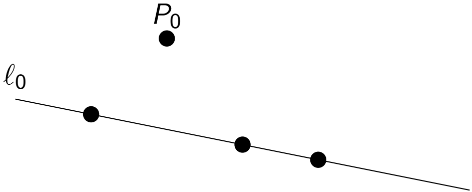
This following was conjectured by J.J Sylvester in 1893, but only definitively proved by Tibor Gallai in the 1930s. The first proof we give is due to L. M. Kelly.

Theorem

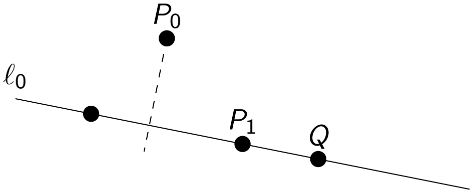
If $n \geq 3$ points in the plane do not lie on a single line, then there is a straight containing exactly two of them.

1st Proof of the theorem

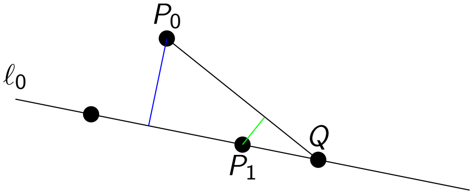
Suppose all lines contain 3 or more points, but that none contains all of them. So for each line ℓ there is a a point P not contained in it. Then there exists a pair $\{\ell_0, P_0\}$ with the shortest perpendicular distance from P_0 to ℓ_0.



Since all lines contain at least 3 points, there are at least 2 on one side of the perpendicular from P_0 to ℓ_0. Of two such points label the one nearest to the perpendicular P_1, and the other, Q.



But the distance from P_1 to P_0Q is less than the distance from P_0 to ℓ_0, and so our assumption that all lines have more than 2 points is wrong.



QED.

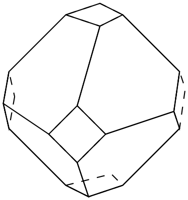
Second proof - using Euler's theorem on polyhedra

Euler's famous theorem on polyhedra relates the numbers of vertices, edges, and faces.

Theorem

Euler's Theorem If v is the number of vertices, e is the number of edges, and f is the number of faces of a polyhedron, not necessarily regular, then:

$$v + f - e = 2$$



Corollary

The average number of edges meeting at any vertex of a polyhedron is strictly less than 6*.*

Proof.

Each faces is bordered by at least three edges, and each edge borders exactly two faces. Therefore

$$2e \geq 3f$$

Substituting this into Euler's theorem, we get

$$v + \frac{2e}{3} \geq e + 2$$

which simplifies to

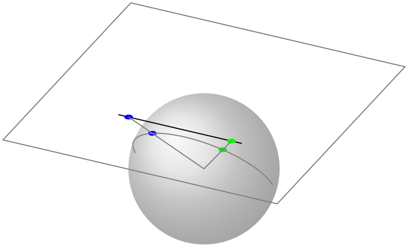
$$3v \geq e + 2 > e$$

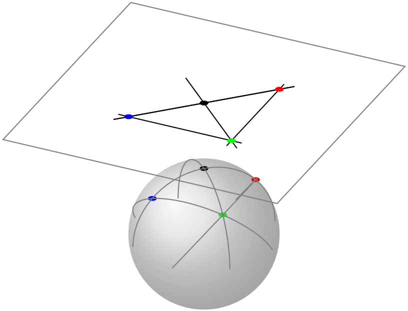
or

$$6 > \frac{2e}{v}$$

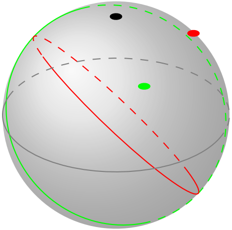
The right-hand side is the average degree since each edge is incident with two vertices.

For this proof we project the points and lines onto the surface of a sphere by joining the points to the centre, and the lines by the intersection of the plane so defined with the sphere. So the straight lines on the plane project to arcs of great circles on the sphere.

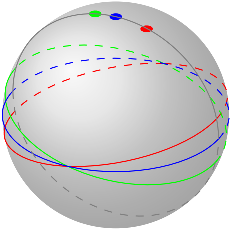




The clever part of this proof is to replace the points by equatorial great circles.



When three or more points are on the same great circle, corresponding to collinear points in the plane, the corresponding equatorial circles intersect at one point.



To complete the proof, we not that the original points and lines in the plane translate to a set of circles and points on the sphere, which forms a polyhedron.
The degree of each point is twice the number of points on each line.
But the average degree of any polyhedron is strictly less than six, so that there must be a line containing only two points!
QED.