Problem 29 from the ex-Cameron set

Oct 2023

This is from a list of problems that Peter says are 'put out to grass' Problem 29. Let n be a positive integer and a a positive real number. It is easy to show that there is a positive real number b (depending on n and a) with the property that, for any positive integers $x1, ..., xn$, if $(1/x1) + ... + (1/xn) < a$, then $(1/x1) + ... + (1/xn) <= a - b$. Problem: If a is an integer, find an explicit lower bound for b in terms of n and a. Having dug around it, I found some quite interesting though not original maths. However, I am at a loss ot see what is unanswered... perhaps you can help!

Firstly the claim: 'It is easy to show that there is a positive real number b (depending on n and a) with the property that, for any positive integers $x1, ..., xn$, if $(1/x1) + ... + (1/xn) < a$, then $(1/x1) + ... + (1/xn) <= a - b$'.
Inductively, if we choose the smallest integer $e1$ such that $1/e1 < a$, then the smallest $e2$ such that $(1/e2) < a - (1/e1)$ and so on until we hace chosen en.
So for any integers $y1, y2, ...yn$ such that
$(1/y1) + (1/y2) + ... + (1/yn) < a$, we know that
$(1/y1) + (1/y2) + ... + (1/yn) <= (1/e1) + ... + (1/en)$
Then we can choose $b = a - (1/e1) + ... + (1/en)$ we have for any
$(1/x1) + ... + (1/xn) < a$,
$(1/y1) + (1/y2) + ... + (1/yn) <= (1/e1) + ... + (1/en) = a - b$.

So limiting a now to be an integer, if $a > 1$, the first $a - 1$ if the e_i would be equal to 1. So from now on assume that $a = 1$.
So clearly $e_1 = 2$ and $e_2 = 3$, and because $1 - (1/2 + 1/3) = (1/6)$, $e_3 = 7$. And since $(1/6) - (1/7) = (1/42)$, $e_4 = 43$, and so on.
So it follows that: $e_n = e_1 e_2 ... e_{n-1} + 1$.

This is known as Sylvester's sequence, A000058 in the Sloane Encyclopedia.

So can we find a formula for e_n? Well, ... sort of but its more interesting than useful! There is a quadratic recurrence relation between the e_i. Since $e_{n-1} = e_1 e_2 e_3 ... e_{n-2} + 1$, we can write $e_n = e_{n-1}(e_{n-1} - 1) + 1 = e_{n-1}^2 - e_{n-1} + 1$.

We can rewrite this as:
$e_n = e_{n-1}^2 - e_{n-1} + 1 = (e_{n-1} - 1/2)^2 + 3/4.$

So $e_n - 1/2 = (e_{n-1} - 1/2)^2 + 1/4.$ and it is convenient to write $f_n = e_n - 1/2$ to get:

$f_{n-1}^2 < f_n = f_{n-1}^2 + 1/4$

Now from this we can show that $\sqrt[2^n]{f_n}$ tends to a finite limit.
From the obvious inequalities
$f_{n-1}^2 < f_n = f_{n-1}^2 + 1/4$ and the fact that $f_i > 1$ for all i,
we get:
$\sqrt[2^{n-1}]{f_{n-1}} < \sqrt[2^n]{f_n} = \sqrt[2^n]{(f_{n-1}^2 + 1/4)}$
So since $f_1 = 3/2$, $f_2 = 5/2$ etc, on the left-hand side we have a monotonic increasing sequence which is bounded below:
$\sqrt{3/2} < \sqrt[4]{5/2} < ... < \sqrt[2^n]{f_n}$.

The right hand side is a little more complicated:
$\sqrt[2^n]{f_n} = \sqrt[2^n]{(f_{n-1}^2 + 1/4)} = \sqrt[2^{n-1}]{(f_{n-1}}(1 + \frac{1}{4f_{n-1}^2})^{\frac{1}{2^n}}$
using the binomial theorem: $(1+x)^{\frac{1}{2^n}} = 1 + \frac{x}{2^n} - .... < 1 + \frac{x}{2^n}$, because the 3rd term is negative

So $\sqrt[2^n]{f_n} < \sqrt[2^{n-1}]{(f_{n-1}} + \frac{\sqrt[2^{n-1}]{(f_{n-1}}}{2^{n+2}f_{n-1}^2} < \sqrt[2^{n-1}]{(f_{n-1}} + \frac{1}{2^{n+2}f_{n-1}}$
which we can iterate into an increasing increasing sequence which his bounded above:

$\sqrt[2^n]{f_n} < \sqrt[2^{n-1}]{f_{n-1}} + \frac{1}{2^{n+2}f_{n-1}} < \sqrt[2^{n-2}]{f_{n-2}} + \frac{1}{2^{n+2}f_{n-1}} + \frac{1}{2^{n+1}f_{n-2}} <$
$... < \sqrt[2^m]{(f_m} + \frac{1}{f_m}\sum_{i=m}^{n-1}(1/2^{i+3}) = \sqrt[2^m]{(f_m} + \frac{1}{2^{m+2}f_m}$

So $\sqrt[2^n]{f_n} - \sqrt[2^m]{f_m} \to 0$ as $m, n \to \infty$, so $\sqrt[2^n]{f_n}$ tends to a finite limit. Let's call it ξ.

Letting $n \to \infty$ in the above the inequality, we can see that:

$$\xi - \frac{1}{2^{m+2} f_m^2} < \sqrt[2^m]{f_m} < \xi \text{ for } m \geq 1.$$

Perhaps surprisingly we can use this to derive an formula for f_n, and thereby e_n.
Raising this to the 2^mth power:
$(\xi - \frac{1}{2^{m+2} f_m})^{2^m} < f_m < \xi^{2^m}$

and using the Binomial theorem to expand the left hand side:

$$\xi^{2^m} - \frac{\xi^{2^m - 1}}{4 f_m} + \ldots < f_m < \xi^{2^m}$$

So

$$0 < \xi^{2^m} - f_m < \frac{\xi^{2^m - 1}}{4 f_m}$$

But how large can this difference get? to fnd out, we have to uilise the above lower limit on f_m:

From $\xi^{2^m} - \frac{\xi^{2^m-1}}{4f_m} + \ldots < f_m$ we can write:

$$\frac{\xi^{2^m-1}}{4f_m} < \frac{\xi^{2^m-1}}{4(\xi^{2^m} - \frac{\xi^{2^m-1}}{4f_m})} = \frac{1}{4\xi(1 - \frac{1}{4\xi f_m})} < \frac{1}{4}$$

using that $f_m \geq f_1 = 3/2$ and $\xi > 1.26$. In fact the largest error is about 0.1

So the final (to me) surprise result, since e_m is an integer:

$e_m = f_m + 1/2 = \lfloor \xi^{2^m} + 1/2 \rfloor$.
This was first proved (I think) by Moshe Vardi in 1991.

One of the open questions on this series is whether they are all square free as appears to be the case. From the definition: $e_n = e_1 e_2 ... e_{n-1} + 1$, it is clear that no two distinct members have a common factor. Although many e_n are prime, many are not. The first non-prime one is $e_5 = 1807 = 13 \times 139$. Many other properties can be found in Sloane.