Convex Polygons in the plane
(by Erdos and Szekeres)

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In 1935 Erdos and Szekeres published the following theorem:

Theorem

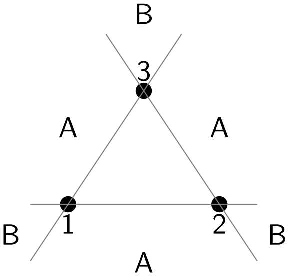
For any given n there exists a number $N(n)$ such that for any set of at least N points in the plane it is possible to find a subset of n points forming a convex polygon.

This was originally conjectured by Esther Klein after she produced a beautiful little proof for the case $n = 4$. The general result is a consequence of Ramsey's theorem, but the authors also produced an alternative proof, which we shall look at here.

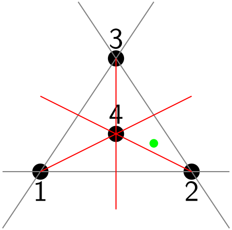
Esther Klein's result

Theorem

Any 5 *points on the plane must contain a convex quadrilateral.*



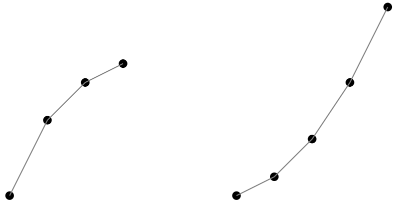
Consider any 3 points forming a triangle. If any fourth point is in one of the regions labelled A, then that would form a convex quadrilateral. If any fourth point is in a B region, exchange it with the nearest triangle point, bringing the triangle point into an enlarged triangle.



So now assuming we have a fourth point inside a triangle, we draw lines through the fourth point and the triangle vertices. Then any fifth point inside the triangle will form a convex quadrilateral with the fourth point and two others, for example the green point with points 1, 2, and 4. QED.

Proof of the general theorem

To begin with , first note that for any finite number of points in the plane there is always a line that is neither parallel nor perpendicular to any of the lines joining any of them, since the number of slopes is also finite. We choose this as the 'horizontal', so that all points are defined by their x and y values relative to this.



Convex Concave

We define a convex path to be one where the slopes are monotonic decreasing, and a concave one where they are monotonic increasing, as in the figure.

Clearly, if we have either a convex set of a concave set of points we can join the endpoints with a straight line to make a convex polygon. So any idea of how many points we need that will guarantee to give us either a convex or a concave set will give us an upper bound on the number we need to give a convex polygon. It won't be a best bound, because for instance most convex polygons would have both a convex and a concave part.

Theorem

Let $f(i,j)$ be the smallest number of points in the plane that must contain either a convex set of length i or a concave set of length j. Then

$$f(i,j) = f(i-1,j) + f(i,j-1) - 1$$

for $i,j > 1$.

It is easy to see that $f(3,n) = f(n,3) = n$

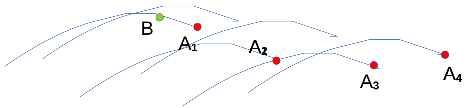
Proof of the convex/concave set theorem

Take $f(i-1,j)+f(i.j-1)-1$ points. first of all we examine the first $f(i-1,j)$ points. If they contain a concave set of length j, then we are done. Otherwise it will contain a convex set of length $i-1$. Now discard the endpoint of this convex set from the $f(i-1,j)$ points and instead include the first of the remaining $f(i,j-1)-1$. So we again have a series of $f(i-1,j)$ points, and this will also contain either a concave set of length j, in which case we are done as before, or a convex set of length $i-1$. If it is the latter, then again discard the endpoint it from the set of $f(i-1,j)$ and add the second of the other $f(i.j-1)-1$ to replace it....and so on.

Assuming there is never a concave set of length j we end up with a series $f(i.j-1)$ endpoints, each the endpoint of a convex set of length $i-1$. This may contain a convex set of length i, in which case we are finished. but otherwise it contains a concave set of length $j-1$.

Call the points of this concave set $A_1, A_2, ...A_{j-1}$. But now consider the point preceding A_1 in the convex set ending in $i-1$, call it B.

If the gradient of BA_1 is less than the gradient of A_1A_2, then $B, A_1, A_2, ...A_{j-1}$ is a concave set of length j. Otherwise the gradient of BA_1 is greater than the gradient of A_1A_2 so A_2 is the endpoint of a convex set of length i.

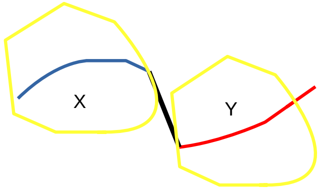


So far we have only proved that

$$f(i,j) \leq f(i-1,j) + f(i,j-1) - 1$$

However, an example shows that this is also sharp: Take a set of points X of size $f(i-1,j)-1$ which contains no convex set of size $i-1$ and no concave set of size j, and set Y of size $f(i,j-1)$ which contains no convex set of size i and no concave set of size $j-1$ and such that all the x coordinates of Y are greater than all those of X. Further, we contrive that all the slopes between a point of X and a point of Y are less than all the slopes between points of X and also less than all the slopes between points of Y, for instance by offsetting the y-values of Y by a suitable amount.

Then $X \cup Y$ is of size $f(i-1,j) + f(i,j-1) - 2$ and contains neither a convex set of size i nor a concave set of size j. This is because although any point of Y will complete a convex set of length $i-1$ in X, no further point of Y can extend it to one of length i. Similarly any point of X can complete a concave set of length $j-1$ in Y, but no other point of X can extend that to one of length j. QED



Using the identity:

$$\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}$$

we can deduce:

$$f(i,j) = \binom{i+j-4}{i-2} + 1$$

The bound $f(n, n) = \binom{2n-4}{n-2}$ gives a correct upper bound for $N(n)$ for $n = 3$, but is already too large for $n = 4$, since $N(4) = 5$ whereas $f(4, 4) = 7$. We know that $N(5) = 9$, but little is known beyond that. Erdos conjectured that $N(n) = 2^{n-2} + 1$, but little progress has been made since!