The Chromatic and Tutte Polynomials

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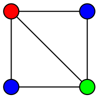
The Chromatic Polynomial

The Chromatic Polynomial is a polynomial in one variable which counts the number of distinct colourings of a graph with a given number of colours. It was introduced by G.D. Birkoff in 1912 for planar graphs while attempting a proof of the 4-colour theorem, and was generalised to all graphs by H. Whitney in 1932.

A proper (vertex) colouring of a graph is a colouring of the vertices such that no two vertices that are joined by an edge have the same colour.



Some examples

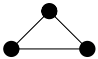
1 Isolated points:

$$P(k) = k^3 \quad (k^n)$$

2 Paths:

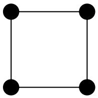
$$P(k) = k(k-1)^2 \quad (k(k-1)^{n-1})$$

3 Complete graphs:



$$P(k) = k(k-1)(k-2) \quad (k!/(k-n)!)$$

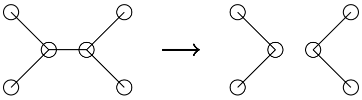
4 Cycles:



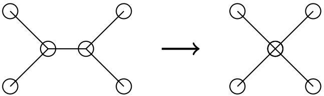
$$P(k) = (k-1)^n + (-1)^n(k-1)$$

Deletion/contraction operations

Two graph operations that are important in graph polynomials are Deletion and contraction, illustrated here:



$G \longrightarrow G \setminus \{e\}$ Deletion



$G \longrightarrow G/\{e\}$ Contraction

The Deletion-Contraction identity for the Chromatic polynomial

Theorem

If P_G, $P_{G\setminus e}$, and $P_{G_{/e}}$ denote the chromatic polynomials for the graph G, G with the edge e deleted, and the edge e contracted respectively, these obey the identity:

$$P_G + P_{G_{/e}} = P_{G\setminus e}$$

Proof.

The colourings of $P_{G\setminus e}$ with k colours can be split into two disjoint sets: those where the end-vertices of e are different and those where the same. The former set is the set of colourings of the whole of G, whereas the latter is those of $P_{G_{/e}}$.

□

The Tutte polynomial

The Tutte polynomial is a two variable graph polynomial. It specialises to the Chromatic polynomial, and to other polynomials that have applications in physics. A wide number of graph statistics can be obtained by setting the variables to particular values.

The Tutte polynomial was devised in 1954. Tutte originally formulated it in terms of spanning trees, but there are two other main constructions: via edge deletion-contraction, and in terms the connected components of edge subsets.

Importantly, whereas we normally consider only simple graphs, the Tutte polynomial also applies to the more general case of graphs with *loops*, i.e. edges on a single vertex, and multiple edges on the same pair of vertices (multigraphs with loops).

Edge Deletion-contraction construction

Definition: A *bridge* is an edge which the only connection between two parts of a graph.

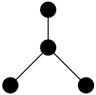
If the Tutte polynomial for a graph G is $T(G, x, y)$, and e is any edge of G:

$$T(G,x,y) = \begin{cases} 1, & \text{if G has no edges} \\ xT(G \setminus \{e\}, x, y), & \text{if e} \quad \text{is a bridge} \\ yT(G/\{e\}, x, y), & \text{if e} \quad \text{is a loop} \\ T(G \setminus \{e\}, x, y) + T(G/\{e\}, x, y), & \text{otherwise} \end{cases}$$

Examples

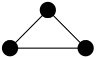
1 Paths, stars:

$$T(G,x,y) = x^2 \quad (x^{|E|} \text{ for a path or star of } |E| \text{ edges})$$



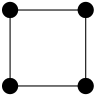
$$T(G,x,y) = x^3$$

2 Triangle:



$$T(G,x,y) = x^2 + x + y$$

3 Cycle:



$$T(G,x,y) = x^3 + x^2 + x + y \quad (\textstyle\sum_{i=1}^{n-1} x^i + y)$$

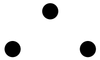
Edge subsets construction

For the graph $G(E, V)$, if A is a set of edges $A \in E$, denote by $k(A)$ the number of connected components of the sub-graph spanned by A.

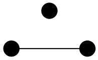
Then $T(G, x, y) = \sum_{\text{all } A \in E} (x-1)^{k(A)-k(E)} (y-1)^{k(A)+|A|-|V|}$

Example -triangle

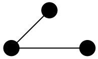
$$k(G) = 1, \quad |V| = 3$$



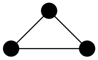
$$A = \emptyset \quad k(A) = 3 : \quad (x-1)^2(y-1)^0 + ...$$



$$|A| = 1 \quad k(A) = 2, \quad 3 \text{ cases:} \quad .. + 3(x-1)^1(y-1)^0 + ..$$



$$|A| = 2 \quad k(A) = 1, \quad 3 \text{ cases:} \quad .. + 3(x-1)^0(y-1)^0 + ..$$



$$|A| = 3 \quad k(A) = 1, \quad 1 \text{ case:} \quad ... + (x-1)^0(y-1)^1$$

So

$$T(G,x,y) = (x-1)^2 + 3(x-1) + 3 + (y-1) = x^2 + x + y$$

as seen before.

Special values of the Tutte polynomial

There are a number of special values of the Tutte polynomial that count comething in a graph. Three of them are to do with sub-forests and spanning forests:

1. $T(G, 2, 1)$ counts the number of sub-forests of G. That is, the number of edge subsets with no cycles.
2. $T(G, 1, 1)$ counts the number of spanning forests, i.e. forests covering all the vertices with the same number of components as G.
3. $T(G, 1, 2)$ counts the number of spanning subgraphs, i.e. subgraphs containing all vertices and with the same number of components as G (i.e. can contain cycles).

$T(G, 2, 2) = 2^{|E|}$, where $|E|$ is the number of edges of G. This is a simple consequenes of the edge-subsets construction above.

Other special values concern orienttions of G, i.e. assignments of directions to each edge.

Tutte polynomial of the complete graph

Calculating $T(G,x,y)$ for a complete graph Z_n laborious! We have already seen that

$$T(Z_3,x,y) = x^2 + x + y$$

It is not too bad to calculate if for Z_4:

$$T(Z_4,x,y) = x^2 + 3x^2 + 2x + 4xy + 2y + 3y^2 + y^3$$

Igor Pak and some earlier authors derived the following recurrence relation:

$$T(Z_{n+1},x,y) = \sum_{k=1}^{n} \binom{n-1}{k-1}(x+y+y^2+ \dots y^{k-1})T(Z_{k-1},x,y)T(Z_{n-k+1},x,y)$$

Tutte also derived a generating function for this, but I have been unable to find an account of it.

References

1 Bela Bollobas: Modern Graph Theory.

2 Wikpedia: The Chromatic Polynomia, The Tutte Polynomial.

3 Igor M Pak, "Computation of the Tutte polynomial of complete graphs",
https://www.math.ucla.edu/ pak/papers/Pak_Computation
_Tutte_polynomial_complete_graphs