Packing $6 \times 2 \times 1$ bricks into a $7 \times 7 \times 7$ box

December 1, 2023

Theorem 0.1. *The maximum number of* $6 \times 2 \times 1$ *bricks that can be packed into a* $7 \times 7 \times 7$ *box is* 27 *(leaving 19 spaces)*

We divide the space of the whole box into $1 \times 1 \times 1$ cubes, with the lower corner of the box ar $(0, 0, 0)$. We represent the cube at coordinates (i, j, k) by the monomial $x^i y^j z^k$, so the whole box is represented by the polynomial:

$$Box(x, y, x) = \sum_{i,j,k=0\ldots 6} x^i y^j z^k = \left(\sum_{i=0}^{i=6} x^i\right)\left(\sum_{j=0}^{j=6} y^j\right)\left(\sum_{k=0}^{k=6} z^k\right)$$

Now the bricks can be in 6 different orientations. We represent a generically oriented brick with its lower corner at $(0, 0, 0)$ by the polynomial

$$Brick_{u,v} = \left(\sum_{i=0}^{i=5} u^i\right)(1 + v)$$

where the length of the brick is along the u dimension , the width along the v dimension, and the thickness (1) along the remaining dimension.

We want to express $Box(x, y, z)$ something of the form

$$\sum_{u,v\in\{x,y,z\},u\neq v} A_{u,v} Brick_{u,v} + C(x, y, z)$$

where the As are all polynomials in x, y, z where all the coefficients are 1s (because no two bricks can occupy the same unit cube. Notice also that Box has the same property). We want C to have the smallest number of terms, so that the bricks occupy the largest possible volume.
This looks difficult in general, but if we just want to know the size of the best packing we can do some substitution to simplify the problem. If we set $y = z = x$, this will preserve the correct number of terms in the polynomial C and the total degree of each term, but not the exact form.
So setting $y = z = x$

$$C(x, x, x) \equiv Box(x, x, x) \mod Brick_{x,x}$$

To simplify the algebra, we can set $S = \left(\sum_{i=0}^{i=5} x^i\right)$, so

$$Brick_{x,x} = S(1 + x)$$

and

$$Box(x,x,x) = \left(S + x^6\right)^3 = S^3 + 3x^6S^2 + 3x^{12}S + x^{18}$$

Now, note that S has a factor of $(1+x)$, i.e. $S = (1+x^2+x^4)(1+x)$, so that $S^2 \equiv 0 \mod Brick_{xx}$.

So the first two terms in the expansion above are composed of complete bricks, leaving only the last two as the remainder:

$$C(x,x,x) = 3x^{12}S + x^{18}$$

There are 19 terms in this remainder. We can't extract any more copies of $S(1+x)$ from this without introducing minus signs, so the smallest hole has size 19.

Since $S = 1 + x + x^2 + x^3 + x^4 + x^5$, the remainder represents 3 lines of 6 unit cubes plus a singleton.

One packing leaves the spaces along three perpendicular edges with the single block in the corner between them.

A remaining question is: do these three lines have to be perpendicular? It is obvious by playing with the bricks that the lines don't all need to be on an outside edge, and that the lines can be broken up by shifting some bricks up and down.

I'm personally convinced that they have to be perpendicular, but I don't know how to prove it (yet).