# Sperner's Lemma and its application

The talk on LSBU seminar
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**Abstract**

This is the note for the talk on LSBU. In this talk, I will present the Sperner's lemma and its application to prove Brouwer's Theorem and Hall's Theorem, as an example of topological method in combinatorics. The main case discussed here is dimension 1 and 2.

## 1 Motivation

Topological Methods in Combinatorics:

Tucker's Lemma → Borsuk-Ulam Theorem('33) → Lovasz-Kneser Theorem('55-'78)

**Theorem 1.1.** *Borsuk-Ulam Theorem*
*For every continuous mapping* $f : S^n \to \mathbb{R}^n$, $\exists x \in S^n$, *s.t* $f(x) = f(-x)$.

Sperner's Lemma('28) → Brouwer Theorem('11)

**Theorem 1.2.** *Brouwer's Theorem*
*Every continuous function* $f : B^n \to B^n$ *of an* $n$*-dimensional ball to itself has a fixed point (a point* $x \in B^n$ *with* $f(x) = x$*).*

Sperner's Lemma → Hall's Theorem('35) → Generalized Hall's Theorem.

**Theorem 1.3.** *Hall's Theorem*
*If* $\forall A \subseteq M$, $|K(A)| \geqslant |A|$, *then there exists a marriage (Matching) of* $M$.

**Theorem 1.4.** *Generalized Hall's Theorem (Aharoni and Haxell 2000)*
*If men know pairs of women, and if If* $\forall A \subseteq M$, $|K(A)| \geqslant 2|A| - 1$, *then there exists a marriage (Matching) of* $M$.

# 2 Trivial case: Dimension one

## 2.1 Sperner's Lemma

**Theorem 2.1.** *Given the interval* $[1, n]$*, we colored "1" with color "Red" and "n" with color "Blue", and other integer points arbitrary by "Red" or "Blue". Then the numbers of interval* $[i, i+1]$ *with both colors are odd.*

*Proof.* By induction on $n$. □

## 2.2 Brouwer's Theorem

**Theorem 2.2.** *Let* $f$ *be a continuous map from* $[a, b]$ *to* $[a, b]$*, i.e,* $a \leqslant f(x) \leqslant b \forall\ x \in [a, b]$*. Then there exists one fixed point, i.e,* $\exists y \in [a, b],\ s.t. f(y) = y$*.*

*Proof.* Let $g(x) = f(x) - x$. Then $g(x)$ is continuous with $g(a) \leqslant 0$ and $g(b) \geqslant 0$. Therefore, there exists at least one point $y \in [a, b]$ s.t. $g(y) = 0$. □

## 2.3 Hall's Theorem

**Theorem 2.3.** *Two men know two women with each men knowing at least one women, then there exists a marriage.*

*Proof.* Proof by A marriage game on simplex. Assume we have two men $M = \{1, 2\}$, two women $W = \{a, b\}$

**Step 1:** Draw $[1, 4]$, then we have point $1, 2, 3, 4$.

**Step 2:** Fixed a map $f : M \to W$. Let $\binom{1}{f(1)}$ be man 1 choose woman $f(1)$. Note $f(1) \in \{a, b\}$. Now we put a pair of couple $\binom{1}{f(1)}$ on point 1 and $\binom{2}{f(2)}$ on point 4. (Note: $f$ is not injective, which means we might choose the same woman for two different men).

**Step 3:** Fixed a map $g : W \to M$, that means choose a man for each woman. and will put some of them in the middle points $2, 3$ by the following rules. (Note: $g$ is not injective, that means we might choose the same man for different woman).

**Step 4:** The rule of putting $\binom{g(a)}{a}$ and $\binom{g(b)}{b}$ on points $2, 3$ is that the pairs between internal point and boundary point should be two different woman. On other words, if $f(1)$ is $a$, we will put $\binom{g(b)}{b}$ on point 2; otherwise $f(1)$ is $b$, then we need to put $\binom{g(a)}{a}$ on point 2. The same for point 3, but note we might put the same pair on 2 and 3 if $f(1) = f(2)$.

**Step 5:** Color the points {1,2,3,4} by the man of the pairs on it, then we know the end point 1 will be colored by 1 and 4 by 1. What's more, the middle points $2, 3$ will color by 1 or 2. From the Sperner's Lemma of dimension one, we know there exists an interval with two different colors.

**Claim:** The interval with two colors is a marriage.

1) If such interval is $[2, 3]$, then from the **Step 3&4**, we know $g$ must be injective, that means $g(a) \neq g(b)$, i.e, $\binom{g(a)}{a}$ and $\binom{g(b)}{b}$ is a marriage. Because if $g(a) = g(b)$, then $2, 3$ will be colored with the same color, say $g(a)$.

2) If such interval is $[1, 2]$, from 1 is colored by 1, we know 2 is colored by 2. On the other hand, from **Step 4** we know that the woman on 1 and 2 are different, therefore, the pairs on 1 and 2 are disjoint, which means it's a marriage.

3) Interval $[3, 4]$ can be discussed similar to $[1, 2]$.
Therefore we complete the proof. □

**Remark 1. Crucial point:** Each internal point link to at most 1 point on boundary. Therefore we can continue in the **step 4**. This property are decided by the simplex at the **Step 1**. As we demonstrated in the seminar, the interval $[1, 3]$ doesn't work as the middle point 2 is connected to both boundary points $1, 3$, then we can't carry on **step 4** if $f(1) \neq f(2)$. Therefore, the proof of the existence of such "NICE" simplex, which is called "**ECONOMICALLY HIERARCHIC**" in ([1]) is a important/difficult point in the whole proof.

# 3 Dimension Two

## 3.1 Sperner's Lemma

**Theorem 3.1.** *Suppose that some "big" triangle with vertices $V_1, V_2, V_3$ is triangulated(that is, decomposed into a finite number of "small" triangles that fit together edge-by-edge.*

*Assume that the vertices in the triangulation get "colors" from the set* $\{1, 2, 3\}$ *such that:*

- *$V_i$ receives color $i$;*

- *Only $i$ and $j$ are used for vertices along the edge from $V_i$ to $V_j$ (for $i \neq j$).*

- *The interior vertices are colored arbitrarily with 1, 2, or 3.*

*Then in the triangulation there must be a small "tricolored" triangle, which has all three different vertex colors.*

*Proof.* **Step 1: Define "doors" in a triangle;**
**Step 2: Consider the triangle with different number of doors: 0, 1, 2;**
**Step 3: Define the paths through the doors, when will it stop;**
**Step 4: By the dimension 1, there are odd numbers of paths.** □

## 3.2 Brouwer's Theorem

Let $\Delta$ be the triangle in $\mathbb{R}^3$ with vertices $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$.

**Theorem 3.2.** *Every continuous map $f : \Delta \to \Delta$ has a fixed point.*

*Proof.* **Step 1**: We use $\delta(\mathcal{T})$ to denote the maximal length of an edge in a triangulation $\mathcal{T}$. Then we can construct an infinite sequence of triangulation $\mathcal{T}_n$ of $\Delta$ such that the sequence of maximal diameters $\delta(\mathcal{T}_n)$ converges to 0. For example, we can let $\mathcal{T}_{n+1}$ be the the barycentric subdivision of $\mathcal{T}_n$.

**Step 2**: Consider the map $f : \Delta \to \Delta$ where $f(x) = (f_1(x), f_2(x), f_3(x))$. We color the vertex $(x_1, x_2, x_3) = x$ with $i$, if $x_i > 0$ and $x_i \geqslant f_i(x)$.

We claim such $i$ must exist.

$$x_1 + x_2 + x_3 = f_1(x) + f_2(x) + f_3(x) = 1 \Rightarrow \exists i,\ s.t.\ x_i \geqslant f_i(x).$$

**Case 1:** If for some $j$, $x_j < f_j(x)$, then $\exists i,\ s.t\ x_i > f_i(x) \geqslant 0$.

**Case 2:** If for no $j$, $x_j < f_j(x)$, then $\forall i, x_i = f_i(x)$. As $x_1 = x_2 = x_3 = 1$, we have $x_i > 0$ for some $i$.

**Step 3** Claim that this coloring satisfies the assumptions of Sperner's Lemma. Consider the fact that the point $(x_1, x_2, x_3)$ associated with color $i$ must satisfies $x_i > 0$.

**Step 4** Sperner's Lemma now tells us that in each triangulation $\mathcal{T}_n$, there exists a tricolored triangle $\{v^{n:1}, v^{n:2}, v^{n:3}\}$ with $\lambda(v^{n:i}) = i$. The sequence of points $\{(v^{n:1})_{n\geqslant 1}\}$ need not converge, but since the simplex $\Delta$ is compact, some subsequence has limited point. W.l.o.g, we can assume $\lim_{n\to\infty} v^{n:1} = x$. Together with $\delta(\mathcal{T}_n) \to 0$, we have

$$\lim_{n\to\infty} v^{n:1} = \lim_{n\to\infty} v^{n:2} = \lim_{n\to\infty} v^{n:3} = x.$$

**Step 5** Since $\forall i, \quad x_i \geqslant f_i(x)$, together with $x_1+x_2+x_3 = f_1(x)+f_2(x)+f_3(x)$, we have $\forall i, \quad x_i = f_i(x)$, which implies $x = f(x)$.

□

### 3.3 Hall's Theorem

## 4 Simplex and high dimension

## 5 Hypergraph

## 6 Dinner

We continue our talk with Brouwer's fixed-point theorem in the dinner. And it's asked by Tony how about the number of fixed points in the Brouwer's theorem as we know the number of full-colored of "small simplex" in Sperner's Lemma is always odd.

This turns out be link with a very deep result in topology: Hopf's index theorem.

In dimension one, the question is the same as how many times a function will come cross the $x$-coordinated if we know $f(a) < 0$ and $f(b) > 0$. Then if counted by the order of the zero points, we know it will always be odd!!!! (Need to caution with the tangent points)

For higher dimension, we have to come to Hopf's theorem, which will need one more note to explain it.

Another problem is that for each continuous map $f : S^1 \to B^1$, there is always a fixed point. (Informally, we can form $S^1$ by identifying the two end points). One possible explanation is that we have $B^1 \hookrightarrow S^1 \to B^1$, but I am not certain with it.

## References

[1] Ron Aharoni and Penny Haxell, Hall's theorem for hypergraphs. J. Graph Theory35 (2000)