Expander Graphs: Introduction

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Nov 2006

Abstract

This note is for the second talk on expanders. But for completeness, it contains the materials used in the first talk as well. We will focus on the definitions, some examples and basic properties.

1 Introduction

Three different views:
1: There is no "bottleneck" in the graph. Cut Size. Manifold.
2: The spectral Gap is bigger.
3: The random walk is quickly convergence.

The problem:Existence is easy, (why?) but the construction is hard. Explicit construction:
1: Margulis;
2: Lubotzky,Philips and Sarnark.
3: Reingold, Vadhan and Wigderson: Zig-zag product. (The motivation for this talk)

Application: (Who cares?) 1: RL=L.
2: PCP. (Dinur's new proof of PCP)

2 The spectrum of a graph

The basic setting: Regular graph G with degree d, $|V(G)| = n, |E(G)| = m$. Three matrixes:
1: Adjacent Matrix: A (Normalized form $\widetilde{A}$)

2: Incident Matrix: K
3: Laplacian Matrix: L (Normalized form $\widetilde{L}$)

Note: It should be noticed that in this note the normalized form and the general form are used rather freely.

The relation:

$$L = KK^t = d - A$$
$$(Af)(x) = \sum_{y \in V} A_{xy} f(y) \text{ for } f \in l^2(V)$$

Some observations:
1: In regular case, A and L gives almost the same information about G.
2: A, L are the symmetric real matrixes. A few facts from the linear algebra.

2.1 The n eigenvalues are real.

2.2 The eigenvectors corresponding to distinct eigenvalues are orthogonal. The eigenvectors corresponding to multiplicative eigenvalues may be orthogonized w.r.t. each other.

2.3 There exists an orthogonal basis forms by n eigenvectors.

2.4 There exists a matrix $P \in SO(n)$, s.t. $P^t AP = diag\{\lambda_1, \cdots, \lambda_n\}$

2.5 Spectral decomposition:

$$A = \lambda_1 P_1 + \cdots + \lambda_n P_n$$

where p_i is the projective operator associate with eigenvalue λ_i. (Think about the Jordan norm form). Let $\{v_1, \cdots, v_n\}$ for the corresponding n eigenvectors which form an orthogonal base for the Euclidean space. Let $\Phi = \{v_1, \cdots, v_n\}$. Then we have $A\Phi = \Phi diag\{\lambda_1, \cdots, \lambda_n\}$, which implies: $\Phi^{-1} A \Phi = diag\{\lambda_1, \cdots, \lambda_n\}$.

The spectrum of a graph G, denoted by *spec* is the spectrum of the matrix $\widetilde{A}$ (or L)

Example:
1: Complete graph
2: Circulant graph (Cycles)
3: Others

Some properties about the spectrum of a graph.

Lemma 2.1. *The eigenvalue of graph G have absolute value at most 1, and G has an eigenvalue of 1 with the uniform vector $\mathbf{u}$ as an eigenvector where $\mathbf{u} = \{1, \cdots, 1\}^T$.*

Proof. Let $\lambda \in \mathrm{Spec}(G)$ and f the associated eigenvector. That is:

$$Af = \lambda f.$$

$f \neq \mathbf{0}$, thus $\exists i$, s.t. $|f_i| = \max_j |f_j| > 0$. W.l.o.g, we assume $i = 1$ here.

$$|\lambda| \cdot |f_1| = |a_{12} f_2 + \cdots + a_{1n} f_n| \leq |f_1| \sum_j |a_{1j}| = |f_1| \tag{1}$$

where the last equality follows from that A is non-negative and doubly stochastic. Thus $|\lambda| \leq 1$.

On the other hand, the equality in Equation 1 always hold for uniform vector $\mathbf{u}$. Therefore, which is a eigenvector for the eigenvalue 1. $\square$

Remark: The equality in Equation 1 holds only if we have either:
1: $\forall j$, $a_{ij} \neq 0 \Rightarrow f_j = f_i$ or
2: $\forall j$, $a_{ij} \neq 0 \Rightarrow f_j = -f_i$. (Bipartite case).

Lemma 2.2. *$\lambda \in \mathrm{Spec}(G) \Rightarrow \lambda^2 \in \mathrm{Spec}(G^2) \Rightarrow p(\lambda) \in \mathrm{Spec}(P(G))$ where p is a polynomial and all the associated eigenvectors are the same.*

Lemma 2.3. *G is connected if and only if the eigenvalue 1 has multiplicity 1.*

Proof. We proof the transposition argument:

G is disconnected $\Leftrightarrow$ 1 occurs with multiplicity ≥ 2 in the spectrum.

"$\Rightarrow$" G disconnected, A is blocked as the follows:

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

where both A_1 and A_2 is a doubly stochastic matrix. Thus it's easy to find two independent eigenvectors. (Locally constant eigenfunctions)

" $\Leftarrow$ " Now assume 1 occurs with multiplicity ≥ 2. Then we know $\exists f \perp \mathbf{u}$ s.t $Af = f$. Let $S = \{i | f_i = \max_j x_j\}$ and $T = V - S$. Then S and T both are not empty. On the other hand, $\lambda = 1$ would implies:

$$\forall\ i\ a_{ij} \neq 0 \Rightarrow f_i = f_j.$$

Therefore:

$$a_{ij} = 0 \ \ \forall i \in S,\ j \in T.$$

Which means S and T are disconnected. $\square$

Remark: # of the occurrence of 1 in Spec(G)= # of components.

Lemma 2.4. *Let G be connected. The following are equivalent:*

i: G is bipartite;

ii: $\mathrm{Spec}(G)$ *is symmetric about 0;*

iii: $\lambda_n = -1$.

Proof. i⇒ii: G bipartite, then A can be blocked as follows:

$$A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$$

Let $f = (f_1, f_2)^T$ be an eigenvector associated with eigenvalue $\lambda \in \mathrm{Spec}(G)$. Then we have:

$$\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} Bf_2 \\ B^T f_1 \end{pmatrix} = \begin{pmatrix} \lambda f_1 \\ \lambda f_2 \end{pmatrix}.$$

Now let $g = (f_1, -f_2)^T$, we have:

$$\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ -f_2 \end{pmatrix} = \begin{pmatrix} -Bf_2 \\ B^T f_1 \end{pmatrix} = \begin{pmatrix} -\lambda f_1 \\ \lambda f_2 \end{pmatrix} = -\lambda \cdot \begin{pmatrix} f_1 \\ -f_2 \end{pmatrix}.$$

Which means g is an eigenvector associated to $-\lambda$. That complete the proof of first step.

ii⇒ iii: This step is clear from Lemma 2.1 and 2.3.

iii⇒ i: Let $f \perp \mathbf{u}$ be the eigenvector associated with -1. That is: $Af = -f$. Let $x \in V$ be such that $|f(x)| = \max_{y \in V} |f(y)|$. Replacing f by $-f$ if necessary, we may assume $f(x) > 0$. Now

$$f(x) = -(Af)(x) = -\sum_{y \in V} a_{xy} f(y) = \sum_{y \in V} a_{xy}(-f(y)). \tag{2}$$

So $f(x)$ is a convex combination of the $-f(y)$'s which are, in modulus, less than $|f(x)|$. Therefore $-f(y) = f(x)$ for every $y \in V$, such that $a_{xy} \neq 0$, that is, for every y adjacent to x. Similarly, if z is a vertex adjacent to any such y, then $f(z) = -f(y) = f(x)$. Define $V_+ = \{y \in V : f(y) > 0\}$ and $V_+ = \{y \in V : f(y) > 0\}$. Then both of such sets are not empty. Because G is connected, this defines a bipartition of G.

□

3 Expanding vs. Spectral Gap

For $F \subseteq V$, the *boundary* ∂F is the set of edges connecting F to $V - F$. (Example)

The *expanding constant*, or *isoperimetric constant* of G, is:

$$h(G) = \inf \left\{ \frac{\partial F}{\min\{|F|, |V - F|\}} : F \subseteq V : 0 < |F| < +\infty \right\}. \tag{3}$$

Remark: History of isoperimetric problem.

Example:

1: The complete graph. If $|F| = l$, then $|\partial F| = l(m - l)$, so that $h(K_n) = n - \lfloor \frac{n}{2} \rfloor \sim \frac{n}{2}$.

2: The cycle C_n. If F is a half-cycle, then ∂F=2, so $h(C_n) \leq \frac{2}{\lfloor n/2 \rfloor} \sim \frac{4}{n}$; in particular, $h(C_n) \to 0$ for $n \to +\infty$.

3: Their spectrum. λ_2.

$$\lambda_2(K_n) = \frac{-1}{n-1};$$

$$\lambda_2(C_n) = \cos \frac{2\pi}{n};$$

Question: The λ_2 is small would implies the "good" expansion property. (Spectral Gap)

Remark $\frac{h(G)}{d}$ is more close to the geometrical meaning, and it can be generalize to the irregular graph.

Theorem 3.1. $\frac{1-\lambda_2}{2} \leq \frac{h(G)}{d} \leq \sqrt{2(1 - \lambda_2)}$.

Question: How to prove directly that bipartite graph is bad for expansion? (This is not true as we will prove later there exists good expanders for random bipartite graphs)The large spectral gap $1 - \lambda_2$ would implies the good expansion, but this gap can't be too big. Roughly speaking,asymptotically we have $\lambda_2 \geq 2\sqrt{k-1}$. One crucial point is that bipartite graphs is bad for mixing, which is controlled by $\lambda(G) = \max(|\lambda_2|, |\lambda_n|)$ rather than $|\lambda_2|$ alone. One way to overcome this problem is considering G^2 rather than G, which has all positive eigenvalues. In this case we have $\lambda(G) = \lambda_2$, which reflects the properties of both expansion and rapidly mixing.

Definition 3.1. $\lambda = \lambda(G) = \max(|\lambda_2|, |\lambda_n|)$.

Remark: we can assume all e/values of G are positive. If not, consider G^2 instead of G.

Definition 3.2. Let $(G_m)_{\geq 1}$ be a family of graphs $G_m = (V_m, E_m)$ indexed by $m \in \mathbb{N}$. Furthermore, fix $d \geq 2$. Such a family $(G_m)_{\geq 1}$ of finite, connected, d-regular graphs is *a family of expanders* if:

1: $|V_m| \to +\infty$ for $m \to +\infty$, and

2: $\exists \epsilon > 0$, s.t. $h(G_m) \geq \epsilon$ for every $m \geq 1$.

Remark:

1: d-regular is crucial here, which means that the number of edges of G_m grows linearly with the number of vertices. Without that assumption, we could just take $G_m = K_m$. (Problem: Too many edges)

2: Condition 2 can be replaced by $1 - \lambda_2(G_m) \geq \epsilon$. That is the algebraic definition. (How about λ_n??)

Main problem: Give explicit construction for families of expanders.

Example: Margulis & LPS construction. (Easy to state, hard to prove).

Example 1. Let $T_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, T_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
The vertices set is: $V_m = \mathbb{Z}_m \times \mathbb{Z}_m$;
The adjacent is defined as follows: if $v = (x, y)^T$, then v is adjacent to $T_1 v$, $T_2 v$, $T_1 v + e_1$ and $T_2 v + e_1$.
This is a family of $(m^2, 8, 5\sqrt{2})$-expanders.

Example 2. A family of 3-regular p-vertex graphs for every prime p. Here $V_p = \mathbb{Z}_p$, and a vertex x is connected to $x+1$, $x-1$ and x^{-1}. (operations are mod$p$, and we define the inverse of 0 to be 0).

3.1 The proof of main theorem

In this section we will prove the main theorem, i.e. Theorem 3.1, which link the spectral gap with the expansion parameter.

Theorem $\frac{1-\lambda_2}{2} \leq \frac{h(G)}{d} \leq \sqrt{2(1-\lambda_2)}$.

Lemma 3.2. Rayleigh quotient.

Let $A \in \mathbb{M}_n(\mathbb{R})$ be symmetric. And Spec$(A) = \{\lambda_1 \geq \cdots \geq \lambda_n\}$ with $\{v_1, \cdots, v_n\}$ be the associated orthogonal eigenvectors basis. For $1 \leq k \leq n$, define $W_k \subsetneq \mathbb{R}^n$ to be $span(v_1, \cdots, v_k)$. Then for all k, we have:

$$x \in W_k - \{0\} \;\Rightarrow\; \frac{< Ax, x >}{< x, x >} \geq \lambda_k; \tag{4}$$

and

$$y \in W_k^{\perp} - \{0\} \;\Rightarrow\; \frac{< Ay, y >}{< y, y >} \leq \lambda_{k+1}. \tag{5}$$

Proof. For $x \in W_k$, we have $x = \sum_{i=1}^{k} c_i v_i$. By orthonormality, we get:

$$\frac{< Ax, x >}{< x, x >} = \frac{\sum_{i=1}^{k} \lambda_i c_i^2}{< x, x >} \geq \frac{\lambda_k \sum_{i=1}^{k} c_i^2}{< x, x >} = \lambda_k. \tag{6}$$

□

Remark: Equality holds exactly when x, y are the eigenvectors corresponding to λ_k, λ_{k+1} respectively.

Lemma 3.3. *For any graph G with normalized adjacency matrix A,*

$$\begin{aligned} \lambda(G) &= \max_{x \perp \mathbf{u}} \frac{| < Ax, x > |}{< x, x >} \\ &= \max_{x \perp \mathbf{u}} \frac{||Ax||}{||x||} \\ &= \max_{x \perp \mathbf{u}} \left| 1 - \frac{< \widetilde{L}x, x >}{< x, x >} \right| \end{aligned}$$

Note: When $\lambda_i \geq 0$, we have $\mu(G) = 1 - \lambda(G) = \mu_2(G)$

Proof.

$$\begin{aligned} | < Ax, x > | &= \left| \sum_{i,j} a_{ij} x_i x_j \right| = \left| \frac{2}{d} \sum_{ij \in E} x_i x_j \right| = \left| ||x||^2 - \frac{2}{d} \left(||x||^2 d - \sum_{ij \in E} 2 x_i x_j \right) \right| \\ &= \left| ||x||^2 - \frac{1}{d} \sum_{ij \in E} (x_i - x_j)^2 \right| \\ &= \left| ||x||^2 - \frac{1}{d} < Lx, x > \right| \end{aligned}$$

where we use the fact that:

$$d \cdot ||x||^2 = \sum_{ij \in E} (x_i^2 + x_i^2),$$

and

$$< Lx, x > = < KK^T x, x > = \sum_{ij \in E} (x_i - x_j)^2.$$

□

Corollary 3.4.

$$1-\lambda(G)=\min\left\{\frac{1}{d}\sum_{ij\in E}(x_i-x_j)^2 \mid \quad \sum_i x_i=0 \ \textit{and} \ \sum_i x_i^2=1\right\}$$

Now we are ready to prove one side of the inequality: $\frac{1-\lambda}{2}\le\frac{h(G)}{d}$ where $\lambda\in\mathrm{Spec}(\widetilde{A})$.

Proof. From the corollary, we only need to construct a vector x such that $\sum_i x_i=0$ and $\sum_i x_i^2=1$ with the properties that $\sum_{ij\in E}(x_i-x_j)^2\le 2h(G)$. One naturally candidate is the vector of the following type:

$$x_i=\begin{cases} a, & \text{if } i\in S,\\ b, & \text{if } i\in\bar{S}.\end{cases}$$

where S is the set achieving $h(G)$. Such a vector satisfies the condition if a and b satisfies the following equations:

$$\begin{aligned} a\cdot|S|+b\cdot|\bar{S}| &= 0\\ a^2\cdot|S|+b^2\cdot|\bar{S}| &= 1\end{aligned}$$

From which we obtain:

$$a=\frac{|\bar{S}|}{\sqrt{|S|\cdot|\bar{S}|\cdot n}},\qquad b=-\frac{|S|}{\sqrt{|S|\cdot|\bar{S}|\cdot n}}$$

Then we have

$$\sum_{ij\in E}(x_i-x_j)^2=E(S,\bar{S})\frac{n}{|S||\bar{S}|}\le 2h(G) \tag{7}$$

since $h(G)=\frac{E(S,|\bar{S}|)}{|S|}$ and $|S|\le n/2$. □

The other side is more difficult and the proof can be found in [1].

3.2 Some properties

The non-constructive proof.

The following lemma, due to Alon and Chung (1998), says that an expander graph behaves like a random graph.

Lemma 3.5. *Let G be a d regular graph with n vertices and set $\lambda=\lambda(G)$ for adjacent matrix A. $\forall S,T\subseteq V$:*

$$\left| |E(S,T) - \frac{d|S||T|}{n} \right| \leq \lambda\sqrt{|S||T|}. \tag{8}$$

Note: $\frac{d|S||T|}{n}$ is the expecting edges between two sets S, T in a random d-regular graph. Here S, T is not need to be disjoint.

Proof. Let 1_S and 1_T be the characteristic vectors of S and T. Expand these vectors in the orthonormal basis of eigenvectors $v_1, \cdots, v_n$, viz, $1_S = \sum_i \alpha_i v_i$ and $1_T = \sum_j \beta_j v_j$. Recall that $v_1 = \mathbf{1}/\sqrt{n}$. Then

$$E(S,T) = 1_S A 1_T = (\sum_i \alpha_i v_i) A (\sum_j \beta_j v_j).$$

Since the v_i are orthonormal eigenvectors, this equals $\sum_i \lambda_i \alpha_i \beta_i$. (It doesn't work for any basis) Since $\alpha_1 =< 1_S, \mathbf{1}/\sqrt{n} >= \frac{|S|}{\sqrt{n}}$, $\beta_1 =< 1_T, \mathbf{1}/\sqrt{n} >= \frac{|T|}{\sqrt{n}}$ and $\lambda_1 = d$:

$$E(S,T) = d\frac{|S||T|}{n} + \sum_{i=2}^{n} \lambda_i \alpha_i \beta_i.$$

By the definition of λ:

$$\left| |E(S,T) - \frac{d|S||T|}{n} \right| = |\sum_{i=2}^{n} \lambda_i \alpha_i \beta_i| \leq \sum_{i=2}^{n} |\lambda_i \alpha_i \beta_i| \leq \lambda \sum_{i=2}^{n} |\alpha_i \beta_i| \tag{9}$$

Finally, by Cauchy-Schwartz:

$$\left| |E(S,T) - \frac{d|S||T|}{n} \right| \leq \lambda ||\alpha||_2 ||\beta||_2 = \lambda ||1_S||_2 ||1_T||_2 = \lambda\sqrt{|S||T|}. \tag{10}$$

□

Remark: We can write $1_S = |S|\mathbf{u} + 1_S^{\perp}$, where the first item "spreads the weight evenly" and $1_S^{\perp}$ is an error term. Such observation is important for us to understand the importance of considering $\lambda(G)$, which is roughly controlling the "error" behavior.

4 Random Walk

Motivation: The set of vertices visited by a length t random walk on an expander graph "looks like" (in some respects) a set of t vertices sampled uniformly and independently.

Definition: Walk, Random Walk, stationary distribution. Bipartite graph.

A vector $\mathbf{p} \in \mathbb{R}^n$ is called a *probability distribution vector* if its coordinates are nonnegative and $\sum_i^n p_i = 1$. (Note: a vertex can be represented by a distribution vector). One important observation is that $p = \mathbf{u} + \sum_{i=2}^n \alpha_i v_i$ where $\{\mathbf{u}, v_2, \cdots, v_n\}$ is the orthonormality base given by A.

Definition 4.1. For any vector $v = \{v_1, \cdots, v_n\}$:

$$||v||_\infty = \max_i |v_i|,$$

$$||v|| = ||v||_2 = \sqrt{\sum_i v_i^2},$$

$$||v||_1 = \sum_i |v_i|.$$

Fact: We have $||v||_\infty \leq ||v|| \leq ||v||_1 \leq \sqrt{n}||v||$.

Definition 4.2. The *mixing time* of a graph G with n vertices is the minimum l such that for all starting distributions π

$$||A^l \pi - \mathbf{u}||_\infty < \frac{1}{2n}. \tag{11}$$

Theorem 4.1. *If G is a connected, d-regular, non-bipartite graph on n vertices, then $\lambda = \lambda(G) \leq 1$ and G has mixing time $O(\frac{\log n}{1-\lambda})$.*

Note: $F = O(G)$ if $F(n)/G(n)$ is bounded above as $n \to \infty$.

Proof. Given any probability distribution π, we have the decomposition

$$\pi = \mathbf{u} + \sum_{i=2}^n \pi_i,$$

where $\pi_i = \alpha_i v_i$ and $\mathbf{u}, v_2, \cdots, v_n$ consists an eigenbase. Now we have:

$$A\pi = \mathbf{u} + \lambda_2 \pi_2 + \cdots + \lambda_n \pi_n.$$

which means

$$\begin{aligned}
||A\pi - \mathbf{u}||^2 &= ||A\mathbf{u} + A\pi_2 + \cdots + A\pi_n - \mathbf{u}||^2 \\
&= ||\lambda_2\pi_2 + \cdots + \lambda_n\pi_n||^2 \text{ since } A\mathbf{u} = \mathbf{u} \\
&= \lambda_2^2||\pi_2||^2 + \cdots + \lambda_n^2||\pi_n||^2 \text{ by the Pythagorean theorem} \\
&\leq \lambda^2(||\pi_2||^2 + \cdots + ||\pi_n||^2) \\
&= \lambda^2(||\pi_2 + \cdots + \pi_n||^2) \\
&= \lambda^2||\pi - \mathbf{u}||^2.
\end{aligned}$$

Thus, each step of the random walk decrease the l^2-distance of the distribution on the vertices to the uniform distance by a factor of at least λ. Therefore, $||A^l\pi - \mathbf{u}|| \leq \lambda^l||\pi - \mathbf{u}||$ for all $l \geq 0$. If follows that:

$$\begin{aligned}
||A^l\pi - \mathbf{u}||_\infty &\leq ||A^l\pi - \mathbf{u}|| \\
&\leq \lambda^l||\pi - \mathbf{u}|| \\
&< \lambda^l||\pi|| \text{ since } \pi - \mathbf{u} \text{ and } \mathbf{u} \text{ are orthogonal} \\
&\leq \lambda^l||\pi||_1 \\
&= \lambda^l.
\end{aligned}$$

It follows that

$$||A^l\pi - \mathbf{u}||_\infty < \frac{1}{2n}$$

when

$$l = O\left(\frac{\log n}{\log\frac{1}{\lambda}}\right) \approx O\left(\frac{\log n}{1-\lambda}\right).$$

To see the this last step, note that

$$\log(1+x) = 1 - \frac{1}{1+x} + O\left(\frac{1}{(1+x)^2}\right)$$

by taking the Taylor expansion of both sides. □

Remark: When G is expanders, then we know λ is bounded by a constant for expanders. Thus, the mixing time on an expander is just $O(\log n)$, which is best possible (up to constant factors) since the diameter of an expander is $O(\log n)$.(Explain it!) It follows that expanders are rapidly mixing, which is used to prove that $RL = L$.

Remark: The mixing time of any d-regular, connected, non-bipartite graph on n vertices is $O(dn^2 \log n)$.

The above theorem depends so heavily on $1 - \lambda$, it's natural to try to bound this quantity for various graphs G. We have already seen that $1 - \lambda = \Omega(1)$ for expanders.

Theorem 4.2. *If G is a connected, d-regular, non-bipartite graph on n vertices, then $1 - \lambda \geq \frac{1}{dn^2}$.*

Proof. We will prove this theorem only for the case when G has only non-negative eigenvalues.

From Corollary 3.4, the spectral gap is given by

$$1 - \lambda(G) = \min \left\{ \frac{1}{d} \sum_{ij \in E} (x_i - x_j)^2 \mid \sum_i x_i = 0 \text{ and } \sum_i x_i^2 = 1 \right\}$$

$\square$

5 Appendix

5.1 The homology interpretation of K

$C_0(G) = \{f : V(G) \to \mathbb{C}\} = l^2(V)$
$C_1(G) = \{f : E(G) \to \mathbb{C}\} = l^2(E)$
Note: $C^i(G)$ is the dual space of $C_i(G)$, and we know there is a natural isomorphism between them: $C_i(G) \cong C^i(G)$. Both ot them are Hilbert space, which have nice base sturcture and inner product.

There are different views for $C_i(G)$: a $\mathbb{C}$-combinations of vertices or edges; a vector in $\mathbb{C}^n$ or $\mathbb{C}^m$; the weighted vertices or edges.

Fix an arbitrary orientation on the edges in G. K is the boundary operator: $K : C_1 \to C_0$. If $e = (xy)$ is an edge from x to y, then $K(e) = y - x$, where e should be understand as a vector $\{0, \cdots, 1, \cdots, 0\} \in \mathbb{C}^m$, the characteristic vector of e.

We have the isomorphic $C_i(G) \cong C^i(G)$, therefore the coboundary operator $K^T : C^0(G) \to C^1(G)$ is the transpoistion of K.

5.2 The second proof of main theorem

Proof. It's better to use the ordinate adjacent matrix here rather than the normalized version. Thus the inequality can be rephrased as:

$$\frac{d - \lambda}{2} \leq h(G)$$

where $\lambda \in \mathrm{Spec}(A)$, which can further simplified as:

$$\lambda \geq d - 2h(G).$$

Therefore the crucial point in this proof is that we should construct a vector $f \perp \mathbf{u}$ such that:

$$\lambda \geq \frac{< Af, f >}{||f||^2} \geq d - 2h(G).$$

Since

$$h(G) = \frac{E(S, \bar{(}S)}{|S|}$$

for some vertices set S where $|S| \leq \frac{n}{2}$, we can construct f as follows.

$$f = |\bar{S}|\mathbf{1}_S - |S|\mathbf{1}_{\bar{S}} = (a, \cdots, a, -b, \cdots, -b),$$

where $a = |\bar{S}|$ and $b = |S|$.

Then we have (in an inform/wrong way):

$$< Af, f >= (a, -b) \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} a \\ -b \end{pmatrix} = a^2 A_{11} + b^2 A_{22} - abA_{12} - abA_{21}.$$

That means

$$< Af, f >= 2(|E(S)|a^2 + E(|\bar{S}|)b^2 - ab|E(S, \bar{S})|).$$

On the other hand, from $a + b = n$ we have:

$$< f, f >= a^2|S| + b^2|\bar{S}| = abn.$$

Since G is d-regular:

$$\begin{aligned} 2|E(S)| &= d|S| - |E(S, \bar{S})| \\ 2|E(\bar{S})| &= d|\bar{S}| - |E(S, \bar{S})| \end{aligned}$$

Then:

$$\begin{aligned} < Af, f > &= a^2(d|S| - |E(S, \bar{S})|) + b^2(d|\bar{S}| - |E(S, \bar{S})|) - 2ab|E(S, \bar{S})| \\ &= d(a^3 + b^3) - (a + b)^2|E(S, \bar{S}| \\ &= d \cdot n \cdot a \cdot b - n^2|E(S, \bar{S})| \end{aligned}$$

The last step is:

$$\begin{aligned} \lambda \geq \frac{< Af, f >}{||f||^2} &= \frac{d \cdot n \cdot a \cdot b - n^2|E(S, \bar{S})|}{n \cdot a \cdot b} \\ &= d - \frac{n \cdot |E(S, \bar{S})|}{|S| \cdot |\bar{S}|} \geq d - 2h(G) \end{aligned}$$

□

References

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