Expander Graphs: Introduction

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Abstract

As the first one of a series talk on Expanders, we will go through some basic setting in this field, which roughly belongs to the Spectral Graph Theory.

1 Introduction

Three different views:
1: There is no "bottleneck" in the graph. Cut Size. Manifold.
2: The spectral Gap is bigger.
3: The random walk is quickly convergence.

The problem:Existence is easy, (why?) but the construction is hard. Explicit construction:
1: Margulis;
2: Lubotzky,Philips and Sarnark.
3: Reingold, Vadhan and Wigderson: Zig-zag product. (The motivation for this talk)

Application: (Who cares?) 1: RL=L.2: PCP. (Dinur's new proof of PCP)

2 The spectrum of a graph

The basic setting: Regular graph G with degree d, $|V(G)| = n, |E(G)| = m$. Three matrixes:
1: Adjacent Matrix: A (Normalized form $\widetilde{A}$)
2: Incident Matrix: K

3: Laplacian Matrix: L (Normalized form $\widetilde{L}$)

The relation:

$$L = KK^t = d - A$$

$$(Af)(x) = \sum_{y \in V} A_{xy} f(y) \text{ for } f \in l^2(V)$$

Some observations:
1: In regular case, A and L gives almost the same information about G.
2: A, L are the symmetric real matrixes. A few facts from the linear algebra.
2.1 The n eigenvalues are real.
2.2 The eigenvectors corresponding to distinct eigenvalues are orthogonal. The eigenvectors corresponding to multiplicative eigenvalues may be orthogonized w.r.t. each other.
2.3 There exists an orthogonal basis forms by n eigenvectors.
2.4 There exists a matrix $P \in SO(n)$, s.t. $P^t AP = diag\{\lambda_1, \cdots, \lambda_n\}$
2.5 Spectral decomposition:

$$A = \lambda_1 P_1 + \cdots + \lambda_n P_n$$

where p_i is the projective operator associate with eigenvalue λ_i. (Think about the Jordan norm form). Let $\{v_1, \cdots, v_n\}$ for the corresponding n eigenvectors which form an orthogonal base for the Euclidean space. Let $\Phi = \{v_1, \cdots, v_n\}$. Then we have $A\Phi = \Phi diag\{\lambda_1, \cdots, \lambda_n\}$, which implies: $\Phi^{-1} A \Phi = diag\{\lambda_1, \cdots, \lambda_n\}$.

The spectrum of a graph G, denoted by *spec* is the spectrum of the matrix $\widetilde{A}$ (or L)

Example:
1: Complete graph
2: Circulant graph (Cycles)
3: Others

Some properties about the spectrum of a graph.

Lemma 2.1. *The eigenvalue of graph G have absolute value at most 1, and G has an eigenvalue of 1 with the uniform vector $\mathbf{u}$ as an eigenvector where $\mathbf{u} = \{1, \cdots, 1\}^T$.*

Proof. Let $\lambda \in \text{Spec}(G)$ and f the associated eigenvector. That is:

$$Af = \lambda f.$$

$f \neq \mathbf{0}$, thus $\exists i$, s.t. $|f_i| = \max_j |f_j| > 0$. W.l.o.g, we assume $i = 1$ here.

$$|\lambda| \cdot |f_1| = |a_{12} f_2 + \cdots + a_{1n} f_n| \leq |f_1| \sum_j |a_{1j}| = |f_1| \tag{1}$$

where the last equality follows from that A is non-negative and doubly stochastic. Thus $|\lambda| \leq 1$.

On the other hand, the equality in Equation 1 always hold for uniform vector $\mathbf{u}$. Therefore, which is a eigenvector for the eigenvalue 1. □

Remark: The equality in Equation 1 holds only if we have either:
1: $\forall j$, $a_{ij} \neq 0 \Rightarrow f_j = f_i$ or
2: $\forall j$, $a_{ij} \neq 0 \Rightarrow f_j = -f_i$. (Bipartite case).

Lemma 2.2. $\lambda \in \mathrm{Spec}(G) \Rightarrow \lambda^2 \in \mathrm{Spec}(G^2) \Rightarrow p(\lambda) \in \mathrm{Spec}(P(G))$ *where p is a polynomial and all the associated eigenvectors are the same.*

Lemma 2.3. *G is connected if and only if the eigenvalue 1 has multiplicity 1.*

Proof. We proof the transposition argument:

G is disconnected $\Leftrightarrow$ 1 occurs with multiplicity ≥ 2 in the spectrum.

"$\Rightarrow$" G disconnected, A is blocked as the follows:

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

where both A_1 and A_2 is a doubly stochastic matrix. Thus it's easy to find two independent eigenvectors. (Locally constant eigenfunctions)

" $\Leftarrow$ " Now assume 1 occurs with multiplicity ≥ 2. Then we know $\exists f \perp \mathbf{u}$ s.t $Af = f$. Let $S = \{i | f_i = \max_j x_j\}$ and $T = V - S$. Then S and T both are not empty. On the other hand, $\lambda = 1$ would implies:

$$\forall\ i\ a_{ij} \neq 0 \Rightarrow f_i = f_j.$$

Therefore:

$$a_{ij} = 0 \ \ \forall i \in S,\ j \in T.$$

Which means S and T are disconnected. □

Remark: # of the occurrence of 1 in Spec(G)= # of components.

Lemma 2.4. *Let G be connected. The following are equivalent:*
i: G is bipartite;
ii: Spec(G) *is symmetric about 0;*
iii: $\lambda_n = -1$.

Proof. i⇒ii: G bipartite, then A can be blocked as follows:

$$A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$$

Let $f = (f_1, f_2)^T$ be an eigenvector associated with eigenvalue $\lambda \in \text{Spec}(G)$. Then we have:

$$\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} Bf_2 \\ B^T f_1 \end{pmatrix} = \begin{pmatrix} \lambda f_1 \\ \lambda f_2 \end{pmatrix}.$$

Now let $g = (f_1, -f_2)^T$, we have:

$$\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ -f_2 \end{pmatrix} = \begin{pmatrix} -Bf_2 \\ B^T f_1 \end{pmatrix} = \begin{pmatrix} -\lambda f_1 \\ \lambda f_2 \end{pmatrix} = -\lambda \cdot \begin{pmatrix} f_1 \\ -f_2 \end{pmatrix}.$$

Which means g is an eigenvector associated to $-\lambda$. That complete the proof of first step.

ii⇒ iii: This step is clear from Lemma 2.1 and 2.3.

iii⇒ i: Let $f \perp \mathbf{u}$ be the eigenvector associated with -1. That is: $Af = -f$. Let $x \in V$ be such that $|f(x)| = \max_{y \in V} |f(y)|$. Replacing f by $-f$ if necessary, we may assume $f(x) > 0$. Now

$$f(x) = -(Af)(x) = -\sum_{y \in V} a_{xy} f(y) = \sum_{y \in V} a_{xy}(-f(y)). \tag{2}$$

So $f(x)$ is a convex combination of the $-f(y)$'s which are, in modulus, less than $|f(x)|$. Therefore $-f(y) = f(x)$ for every $y \in V$, such that $a_{xy} \neq 0$, that is, for every y adjacent to x. Similarly, if z is a vertex adjacent to any such y, then $f(z) = -f(y) = f(x)$. Define $V_+ = \{y \in V : f(y) > 0\}$ and $V_+ = \{y \in V : f(y) > 0\}$. Then both of such sets are not empty. Because G is connected, this defines a bipartition of G.

□

3 Expanding vs. Spectral Gap

For $F \subseteq V$, the *boundary* ∂F is the set of edges connecting F to $V - F$. (Example)

The *expanding constant*, or *isoperimetric constant* of G, is:

$$h(G) = \inf \left\{ \frac{\partial F}{\min\{|F|, |V - F|\}} : F \subseteq V : 0 < |F| < +\infty \right\}. \tag{3}$$

Remark: History of isoperimetric problem.

Example:

1: The complete graph. If $|F| = l$, then $|\partial F| = l(m-l)$, so that $h(K_n) = n - \lfloor \frac{n}{2} \rfloor \sim \frac{n}{2}$.

2: The cycle C_n. If F is a half-cycle, then ∂F=2, so $h(C_n) \leq \frac{2}{\lfloor n/2 \rfloor} \sim \frac{4}{n}$; in particular, $h(C_n) \to 0$ for $n \to +\infty$.

3: Their spectrum. λ_2.

$$\lambda_2(K_n) = \frac{-1}{n-1};$$

$$\lambda_2(C_n) = \cos\frac{2\pi}{n};$$

Question: The λ_2 is small would implies the "good" expansion property. (Spectral Gap)

Remark $\frac{h(G)}{d}$ is more close to the geometrical meaning, and it can be generalize to the irregular graph.

Theorem 3.1. $\frac{1-\lambda_2}{2} \leq \frac{h(G)}{d} \leq \sqrt{2(1-\lambda_2)}$.

Question: How to prove directly that bipartite graph is bad for expansion?

Definition 3.1. $\lambda = \lambda(G) = \max(|\lambda_2|, |\lambda_n|)$.

Remark: we can assume all e/values of G are positive. If not, consider G^2 instead of G.

Definition 3.2. Let $(G_m)_{\geq 1}$ be a family of graphs $G_m = (V_m, E_m)$ indexed by $m \in \mathbb{N}$. Furthermore, fix $d \geq 2$. Such a family $(G_m)_{\geq 1}$ of finite, connected, d-regular graphs is *a family of expanders* if:

1: $|V_m| \to +\infty$ for $m \to +\infty$, and

2: $\exists \epsilon > 0$, s.t. $h(G_m) \geq \epsilon$ for every $m \geq 1$.

Remark:

1: d-regular is crucial here, which means that the number of edges of G_m grows linearly with the number of vertices. Without that assumption, we could just take $G_m = K_m$. (Problem: Too many edges)

2: Condition 2 can be replaced by $1 - \lambda_2(G_m) \geq \epsilon$. That is the algebraic definition.

Main problem: GIve explicit construction for families of expanders.

Example: Margulis & LPS construction. (Easy to state, hard to prove).

Example 1. Let $T_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, T_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
The vertices set is: $V_m = \mathbb{Z}_m \times \mathbb{Z}_m$;
The adjacent is defined as follows: if $v = (x, y)^T$, then v is adjacent to $T_1 v$, $T_2 v$, $T_1 v + e_1$ and $T_2 v + e_1$.
This is a family of $(m^2, 8, 5\sqrt{2})$-expanders.

Example 2. A family of 3-regular p-vertex graphs for every prime p. Here $V_p = \mathbb{Z}_p$, and a vertex x is connected to $x+1$, $x-1$ and x^{-1}.

3.1 Main theorem proof

Lemma 3.2. Rayleigh quotient.

Let $A \in \mathbb{M}_n(\mathbb{R})$ be symmetric. And $\mathrm{Spec}(A) = \{\lambda_1 \geq \cdots \geq \lambda_n\}$ with $\{v_1, \cdots, v_n\}$ be the associated orthogonal eigenvectors basis. For $1 \leq k \leq n$, define $W_k \subsetneq \mathbb{R}^n$ to be $span(v_1, \cdots, v_k)$. Then for all k, we have:

$$x \in W_k - \{0\} \;\Rightarrow\; \frac{< Ax, x >}{< x, x >} \geq \lambda_k; \tag{4}$$

and

$$y \in W_k^{\perp} - \{0\} \;\Rightarrow\; \frac{< Ay, y >}{< y, y >} \leq \lambda_{k+1}. \tag{5}$$

Proof. For $x \in W_k$, we have $x = \sum_{i=1}^{k} c_i v_i$. By orthonormality, we get:

$$\frac{< Ax, x >}{< x, x >} = \frac{\sum_{i=1}^{k} \lambda_i c_i^2}{< x, x >} \geq \frac{\lambda_k \sum_{i=1}^{k} c_i^2}{< x, x >} = \lambda_k. \tag{6}$$

□

Remark: Equality holds exactly when x, y are the eigenvectors corresponding to λ_k, λ_{k+1} respectively.

Lemma 3.3. *For any graph G with normalized adjacency matrix A,*

$$\begin{aligned} \lambda(G) &= \max_{x \perp \mathbf{u}} \frac{| < Ax, x > |}{< x, x >} \\ &= \max_{x \perp \mathbf{u}} \frac{||Ax||}{||x||} \\ &= \max_{x \perp \mathbf{u}} \left| 1 - \frac{< \widetilde{L}x, x >}{< x, x >} \right| \end{aligned}$$

Note: When $\lambda_i \geq 0$, we have $\mu(G) = 1 - \lambda(G) = \mu_2(G)$

Proof.

$$
\begin{aligned}
| < Ax, x > | &= \left| \sum_{i,j} a_{ij} x_i x_j \right| = \left| \frac{2}{d} \sum_{ij \in E} x_i x_j \right| = \left| ||x||^2 - \frac{2}{d} \left(||x||^2 d - \sum_{ij \in E} 2 x_i x_j \right) \right| \\
&= \left| ||x||^2 - \frac{1}{d} \sum_{ij \in E} (x_i - x_j)^2 \right| \\
&= \left| ||x||^2 - \frac{1}{d} < Lx, x > \right|
\end{aligned}
$$

where we use the fact that:

$$
d \cdot ||x||^2 = \sum_{ij \in E} (x_i^2 + x_i^2),
$$

and

$$
< Lx, x > = < KK^T x, x > = \sum_{ij \in E} (x_i - x_j)^2.
$$

□

4 Appendix

4.1 The homology interpretation of K

$C_0(G) = V(G), C_1(G) = E(G)$ The simplex. The chain.
$C^0(G) = \{f : V(G) \to \mathbb{C}\} = l^2(V)$
$C^1(G) = \{f : E(G) \to \mathbb{C}\} = l^2(E)$
Note: $C^i(G)$ is a Hilbert space: we can define the inner product. Have nice bases. The function can be views as a vector in $\mathbb{C}^n$. We have the isomorphic $C_i(G) \cong C^i(G)$, therefore the coboundary operator is the transpoistion of K.

$$
(Af)(x) = \sum_{y \in V} A_{xy} f(y).
$$

A is a operators on $l^2(V)$.

K is the boundary operator: $K : C_1 \to C_0$. If $e = (xy)$ is a directed edge from x to y, then $K(e) = y - x$ where e is coordinated in edges space.
K^t is the coboundary operator: $K : C^0 \to C^1$.