THREE PRIMES

The Hardy–Littlewood circle method is used to prove Vinogradov's theorem: every sufficiently large odd integer is the sum of three primes

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Background

We shall closely follow *Modern Prime Number Theory* by T. Estermann, CUP, 1961.

We adopt the convention that the variable p with or without a subscript always ranges over the primes. Let us also get some function definitions out of the way:

$$\begin{aligned}
e(x) &= e^{2\pi i x}; \\
\mu(v) &= \left\{ \begin{array}{ll} \prod_{p|v}(-1) & \text{if } v \text{ is square-free,} \\ 0 & \text{otherwise} \end{array} \right\}, && \text{the Möbius function;} \\
\phi(v) &= \sum_{0<h\le v,\ \gcd(h,v)=1} 1, && \text{Euler's phi function;} \\
c_q(v) &= \sum_{0<h\le q,\ \gcd(h,q)=1} e\left(\frac{hv}{q}\right), && \text{Ramanujan's sum.}
\end{aligned}$$

Observe that $c_q(v)$ is just the sum of the v-th powers of the primitive q-th roots of 1.

Lemma 1 *(i) For any X and any integer v, $\int_X^{X+1} e(vx)dx = 0$ if $v \ne 0$ and 1 if $v = 0$.*
(ii) The Möbius function and Euler's phi function are multiplicative. Ramanujan's sum is multiplicative over q.
(iii) If $\gcd(v, q) = 1$, then $c_q(v) = c_q(1)$.
(iv) If $\gcd(v, q) = 1$, then $c_q(v) = \mu(q)$.

Proof Property (i) is fundamental to a lot of what follows. Its proof is straightforward.
If $\gcd(q_1, q_2) = 1$, then

$$c_{q_1}(v)c_{q_2}(v) = \sum_{h_1,h_2} e\left(v\left(\frac{h_1}{q_1} + \frac{h_2}{q_2}\right)\right) = \sum_{h_1,h_2} e\left(\frac{v(h_1q_2 + h_2q_1)}{q_1q_2}\right) = c_{q_1q_2}(n)$$

since $e\left(\frac{h_1q_2 + h_2q_1}{q_1q_2}\right)$ runs over the primitive (q_1q_2)-th roots of 1. That takes care of Ramanujan's sum. The other two are elementary number theory. This proves (ii).
Part (iii) is obvious (I think). For (iv), we can assume $v = 1$ by (iii). If $k \ge 1$, $c_{p^k}(1)$ is the sum of the (p^k)-th roots of 1 minus the sum of the (p^{k-1})-th roots of 1; that is, -1 when $k = 1$ and 0 otherwise. Part (iv) follows by multiplicativity. □

In its simplest form the prime number theorem states that $\pi(x) \sim x/(\log x)$. For a more accurate version we define the *logarithmic sum*,

$$\operatorname{ls}(x) = \sum_{2\le m\le x} \frac{1}{\log m}.$$

This is like the logarithmic integral except that it is a sum, and it is easily seen that the difference between the two is bounded: $\operatorname{ls}(x) - \operatorname{li}(x) = O(1)$.

Theorem 1 (T19 : Theorem 19 in Estermann's book)

$$\pi(x) \;=\; \mathrm{ls}(x) + O\left(x \exp\left(-\frac{\sqrt{\log x}}{200}\right)\right).$$

This (with $\mathrm{li}(x)$ instead of $\mathrm{ls}(x)$) was proved by de la Vallée Poussin in 1898ish. Like the $\mathrm{li}(x)$ form it is a very good approximation. That complicated error term is ultimately superior to $x/(\log x)^m$ for any fixed positive m but it is worse than $x^{1-\delta}$ for any fixed $\delta > 0$ however small. The best error term for the prime number theorem is still the 45-year-old result of H.-E. Richert (1967). There exists a positive constant C_1 such that

$$\pi(x) \;=\; \mathrm{li}(x) + O\left(x \exp\left(-\frac{C_1\,(\log x)^{3/5}}{(\log\log x)^{1/5}}\right)\right),$$

proved by establishing that the Riemann zeta function $\zeta(\sigma + it)$ has no zeros with $\sigma \geq 1 - C_2(\log t)^{-2/3}(\log\log t)^{-1/3}$ for some positive constant C_2. The Holy Grail of the subject is of course to extend the zero-free region westwards all the way up to the line $\sigma = 1/2$, with the consequent improvement in the error term of Theorem 1 to $O(\sqrt{x}\log x)$. This is the Riemann Hypothesis. In the other direction Littlewood showed that

$$\pi(x) - \mathrm{li}(x) \;=\; \Omega_{\pm}\left(\frac{x^{1/2}\log\log x}{\log x}\right).$$

We will also need the prime number theorem for arithmetic progressions.

Theorem 2 (T55) *Let $u > 0$. Let $q \leq (\log x)^u$ and $\gcd(h,q) = 1$. Then the number of primes $p \leq x$, $p \equiv h \pmod{q}$ is given by*

$$\pi(x;q,h) \;=\; \frac{\mathrm{ls}(x)}{\phi(q)} + O\left(x \exp\left(-\frac{\sqrt{\log x}}{200}\right)\right),$$

where the constant implied by the O notation is independent of q and h.

The proof requires the severe restriction on q to guarantee uniformity with respect to q, which is vital for our application. Uniformity with respect to h is 'trivial'. A long-standing problem in this area is to extend the range of q. Write

$$E(x,q) \;=\; \max_{\gcd(h,q)=1}\left|\pi(x;q,h) - \frac{\pi(x)}{\phi(q)}\right|.$$

Then Theorems 1 and 2 state that for $q \leq (\log x)^{15}$, say,

$$E(x,q) \;=\; O\left(x \exp\left(-\frac{\sqrt{\log x}}{200}\right)\right).$$

The Elliott–Halberstam conjecture is that *on average* one can relax the condition on q: for every $\theta < 1$ and $A > 0$ there exists a constant $C_3 > 0$ such that

$$\sum_{1\leq q\leq x^{\theta}} E(x,q) \;\leq\; \frac{C_3 x}{(\log x)^A}.$$

Bombieri and Vinogradov showed that the Elliott–Halberstam conjecture holds for $\theta < 1/2$.

Three primes

Vinogradov proved in 1937 that every sufficiently large odd integer can be represented in the form $p_1 + p_2 + p_3$. Previously, in 1923, Hardy and Littlewood had shown that this is true if there exists a number $\delta < 3/4$ such that none of Dirichlet's L-functions has zeros in the half-plane $\Re z > \delta$. More recently, in 1989 Chen & Wang showed that the three primes representation holds (unconditionally) for odd $n > 10^{43000}$.

Let $r(n)$ denote the number of solutions of $n = p_1 + p_2 + p_3$; that is,

$$r(n) \;=\; \sum_{\substack{n = p_1+p_2+p_3 \\ \text{prime } p_1,p_2,p_3 \geq 2}} 1.$$

Repetitions are allowed and order is relevant; so $r(11) = 6$ because $11 = 2+2+7 = 2+7+2 = 3+3+5 = 3+5+3 = 5+3+3 = 7+2+2$. Let

$$\rho(n) \;=\; \sum_{\substack{n = m_1+m_2+m_3 \\ m_1,m_2,m_3 \geq 2}} \frac{1}{\log m_1}\,\frac{1}{\log m_2}\,\frac{1}{\log m_3}.$$

Ultimately we want the following.

Theorem 3 *Let*

$$S(n) \;=\; \sum_{q=1}^{\infty} \frac{\mu(q)}{\phi^3(q)}\, c_q(n).$$

Then

$$r(n) \;=\; S(n)\rho(n) + O\left(\frac{n^2}{(\log n)^4}\right).$$

The proof will occupy the next three sections of these Notes.

For now we observe that the thing being summed in Theorem 3, $\mu(q)c_q(n)/\phi^3(q)$, is multiplicative as a function of q (see Lemma 1). Moreover, $\mu(1) = 1$, $\mu(p) = -1$ and $\mu(p^k) = 0$ for $k \geq 2$. So $S(n)$ has the simple product form:

$$S(n) \;=\; \prod_p \left(1 - \frac{c_p(n)}{(p-1)^3}\right).$$

If n is even, then $c_2(n) = e(n/2) = 1$, $S(n) = 0$, and Theorem 3 doesn't say anything interesting. On the other hand, when n is odd we have $c_2(n) = -1$, $|c_p(n)| \leq p - 1$ for odd p and hence

$$S(n) \;=\; 2\prod_{p>2}\left(1 - \frac{c_p(n)}{(p-1)^3}\right) \;\geq\; 2\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right) \;\geq\; 2\prod_{m=2}^{\infty}\left(1 - \frac{1}{m^2}\right) \;=\; 1.$$

Furthermore we can estimate $\rho(n)$. For $n \geq 6$, the number of terms in the sum for $\rho(n)$ is $\displaystyle\sum_{m_1=2}^{n-4}(n - m_1 - 3) = \frac{1}{2}(n-4)(n-5)$ and each term is at least $1/(\log n)^3$. Therefore

$$\rho(n) \;>\; \frac{1}{(\log n)^3}\sum_{m_1=2}^{n-4}(n - m_1 - 3) \;>\; \frac{n^2}{3(\log n)^3}$$

for sufficiently large n. Thus we have our desired result:

$$r(n) \;>\; \frac{n^2}{3(\log n)^3} + O\left(\frac{n^2}{(\log n)^4}\right).$$

The computation of $r(n)$

For $v \geq 0$, let

$$f(x,v) \;=\; \sum_{p \leq v} e(px).$$

Then

$$f(x,v)^3 \;=\; \sum_{p_1 \leq v} \sum_{p_2 \leq v} \sum_{p_3 \leq v} e((p_1+p_2+p_3)x)$$

and

$$r(n) \;=\; \int_{x_0}^{x_0+1} f(x,n)^3 e(-nx)dx. \tag{1}$$

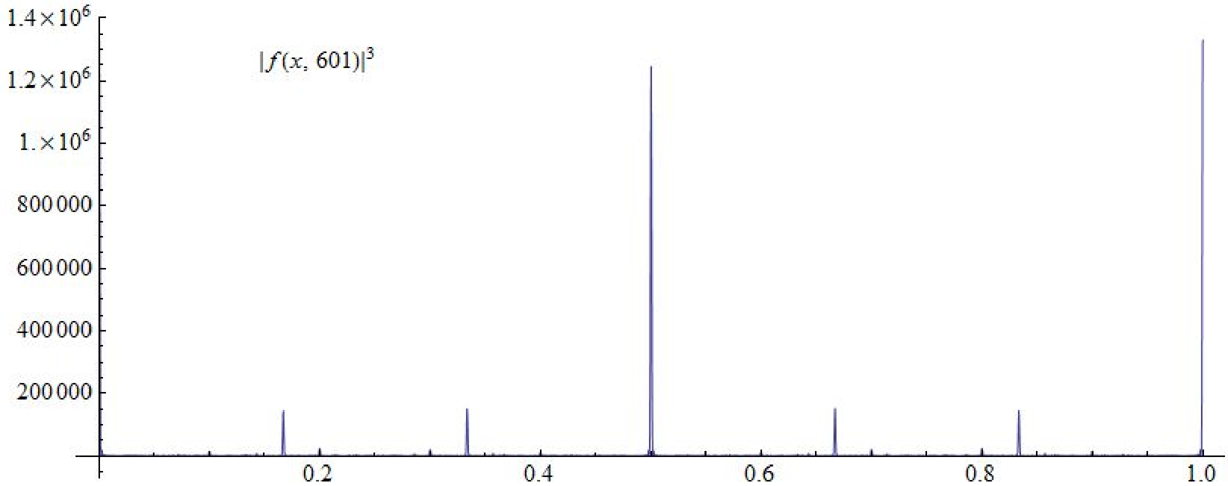
Curiously, this formula actually works, at least for small n. Putting it into MATHEMATICA gives this table.

n	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$r(n)$	1	3	3	4	6	6	9	6	6	10	9	12	12	12	12	19	12	21	15	21	18	30	15	30	12

But our main task is to find a non-trivial general lower bound for $r(n)$ by proving Theorem 3. We shall estimate the integral in (1). Henceforth we will assume tacitly that n is sufficiently large. For convenience we fix

$$x_0 \;=\; \frac{(\log n)^{15}}{n},$$

which we want to be small—and it will be provided n is sufficiently large.



It happens that $f(x,n)^3$ is small unless x is near a rational number with a small square-free denominator. Even for the tiny value $n = 601$ we can clearly see the spikes at 0, 1/2, 1/3, 2/3, 1/6 and 5/6 as well as lesser peaks at $j/10$ for $j = 1, \ldots, 9$, $j \neq 5$; but there are none at 1/4, 3/4, 1/8, 3/8, 5/8 and 7/8. It turns out that the half-width of the six most prominent spikes is about 0.001, and if we integrate $f(x,601)^3 e(-nx)$ over just the intervals $a \pm 0.001$, $a = 0$, 1/2, 1/3, 2/3, 1/6, 5/6, we obtain 2766, compared with the true value $r(601) = 2835$.

We split the interval $[x_0, x_0+1]$ into *major arcs*, and *minor arcs*. The major arcs will consist of all those numbers that are within x_0 of a rational number with a denominator not exceeding $(\log n)^{15}$. The minor arcs consist of everything else in $[x_0, x_0+1]$; as we shall see later (Theorems 6 and 7) these numbers are always within x_0 of a rational number with a denominator in the range $\big((\log n)^{15},\ n/(\log n)^{15}\big)$. We attempt to find a reasonably accurate estimate of the integral (1) for $r(n)$ over the major arcs, where we often expect $f(x,n)$ to be large. On the minor arcs we are content to find a non-trivial upper bound for $f(x,n)$.

For a typical major arc, we write

$$\mathcal{J}(h,q) \;=\; \int_{h/q-x_0}^{h/q+x_0} f(x,n)^3 e(-nx)dx.$$

We can assume n is so large that the major arcs do not overlap. Hence

$$r(n) \;=\; \sum_{q\le(\log n)^{15}} \sum_{\substack{0<h\le q\\ \gcd(h,q)=1}} \mathcal{J}(h,q) + \int_{\text{minor arcs}} f(x,n)^3 e(-nx)dx.$$

The minor arcs

We estimate the integral over the minor arcs with the next few theorems, beginning with an equality which says that adding something small to x hopefully won't change $|f(x,v)|$ too much.

Theorem 4 (151 : Estermann (151)) *The function $f(x,v)$ satisfies the identity*

$$f(x+y,v) \;=\; e(vy)f(x,v) - 2\pi i y \int_0^v e(uy)f(x,u)du.$$

Proof Observe that

$$e(vy) - e(wy) \;=\; 2\pi i y \int_w^v e(uy)du.$$

Hence

$$\begin{aligned} f(x+y,v) &= \sum_{p\le v} e(px)e(py) \;=\; \sum_{p\le v} e(px)\left(e(vy) - 2\pi i y\int_p^v e(uy)du\right) \\ &= e(vy)\sum_{p\le v} e(px) - 2\pi i y\int_0^v e(uy)\sum_{p\le u} e(px)du \\ &= e(vy)f(x,v) - 2\pi i y\int_0^v e(uy)f(x,u)du. \end{aligned}$$

□

Theorem 5 (T56) *Let*

$$(\log n)^{15} \;<\; q \;\le\; \frac{n}{(\log n)^{15}}, \qquad \gcd(h,q) \;=\; 1.$$

and $v \le n$. Then

$$\left|f\left(\frac{h}{q}, v\right)\right| \;\le\; \frac{n}{(\log n)^3}.$$

Proof Theorem 5 is the vital minor arcs estimate that makes the circle method work for the three primes problem. This is where Vinogradov succeeded after H & L failed. The proof is elementary but complicated and is omitted. See Estermann, pp 54–61. □

Theorem 6 (T57) *Given any x and any $y \geq 1$, there exist h and q with $q \leq y$ and* $\gcd(h, q) = 1$ *such that*

$$|qx - h| \ < \ \frac{1}{y}.$$

Proof We may assume $0 < x < 1$. Let $m = \lfloor y \rfloor$. Then

$$x \ \in \ \left[\frac{h_1}{q_1}, \ \frac{h_1 + h_2}{q_1 + q_2}\right) \qquad \text{or} \qquad x \ \in \ \left[\frac{h_1 + h_2}{q_1 + q_2}, \ \frac{h_2}{q_2}\right),$$

where h_1/q_1 and h_2/q_2 are consecutive fractions in the Farey sequence of order m; so that $h_2 q_1 - h_1 q_2 = 1$ and $q_1 + q_2 \geq m + 1$. If x is in the first interval, we take $h/q = h_1/q_1$ since

$$x - \frac{h_1}{q_1} \ < \ \frac{h_2 q_1 - h_1 q_2}{q_1(q_1 + q_2)} \ = \ \frac{1}{q_1(q_1 + q_2)} \ \leq \ \frac{1}{q_1(m + 1)} \ \leq \ \frac{1}{q_1 y}.$$

Similarly we take $h/q = h_2/q_2$ if x is in the second interval. □

Theorem 7 (152) *Suppose x is in a minor arc. Then*

$$|f(x, n)| \ = \ O\left(\frac{n}{(\log n)^3}\right).$$

Proof Suppose x is in a minor arc. Then by Theorem 6 with $y = n/(\log n)^{15}$ there exist coprime h and q with

$$(\log n)^{15} \ < \ q \ < \ \frac{n}{(\log n)^{15}} \qquad \text{and} \qquad |qx - h| \ < \ \frac{(\log n)^{15}}{n} \ = \ x_0,$$

the first inequality because otherwise x would be in a major arc. Therefore by Theorem 5

$$\left| f\left(\frac{h}{q}, v\right) \right| \ \leq \ \frac{n}{(\log n)^3}, \qquad 0 \leq v \leq n.$$

Putting $z = x - h/q$ and using Theorem 4, we have

$$\begin{aligned} |f(x, n)| \ &= \ \left| f\left(\frac{h}{q} + z, n\right) \right| \ = \ \left| e(nz) f\left(\frac{h}{q}, n\right) - 2\pi i z \int_0^n e(uz) f\left(\frac{h}{q}, u\right) du \right| \\ &\leq \ \frac{n}{(\log n)^3} + 2\pi z \int_0^n \frac{n}{(\log n)^3} du \\ &\leq \ \frac{(1 + 2\pi) n}{(\log n)^3} \end{aligned}$$

since $|z| = |x - h/q| < x_0/q < 1/n$. □

We are now ready to establish the desired non-trivial upper bound for the $r(n)$ integral (1) over the minor arcs.

Theorem 8 (202) *We have*

$$\int_{\text{minor arcs}} f(x, n)^3 e(-nx) dx \ = \ O\left(\frac{n^2}{(\log n)^4}\right).$$

Proof By Theorem 7,

$$\int_{\text{minor arcs}} f(x,n)^3 e(-nx)dx \;=\; O\left(\frac{n}{(\log n)^3}\int_0^1 |f(x,n)|^2 dx\right).$$

But

$$\int_0^1 |f(x,n)|^2 dx \;=\; \sum_{p_1\le n}\sum_{p_2\le n}\int_0^1 e((p_1-p_2)x)dx \;=\; \sum_{p\le n} 1 \;=\; \pi(n)$$

and the result follows from the prime number theorem (Theorem 1). □

The major arcs

Let

$$g(x,v) \;=\; \begin{cases} \displaystyle\sum_{2\le m\le v} \frac{e(mx)}{\log m} & \text{for } v\ge 2,\\ 0 & \text{for } v<2.\end{cases}$$

So $g(x,v)$ is like $f(x,v)$ but instead of summing over primes we sum over integers m with weight $1/(\log m)$ to account for the approximate density of the primes at m.

Theorem 9 (204) *The function $g(x,v)$ satisfies the identity*

$$g(x+y,v) \;=\; e(vy)g(x,v) - 2\pi i y\int_0^v e(uy)g(x,u)du.$$

Proof Similar to Theorem 4. □

Theorem 10 (T58) *Suppose*

$$q\le(\log n)^{15},\qquad \gcd(h,q)=1,\qquad \text{and}\qquad |y|\le x_0=\frac{(\log n)^{15}}{n}.$$

Then

$$\left|f\left(\frac{h}{q}+y,n\right)-\frac{\mu(q)}{\phi(q)}g(y,n)\right| \;\le\; \frac{n}{(\log n)^{69}}.$$

Proof Suppose $v\le n$. We have

$$\left|f\left(\frac{h}{q},v\right)-\sum_{p\le v,\ p\nmid q} e\left(\frac{ph}{q}\right)\right| \;\le\; \sum_{p|q}1 \;<\; q.$$

But

$$\sum_{p\le v,\ p\nmid q} e\left(\frac{ph}{q}\right) \;=\; \sum_{\substack{0<l\le q\\ \gcd(l,q)=1}} e\left(\frac{lh}{q}\right)\sum_{\substack{p\le v\\ p\equiv l \pmod q}} 1 \;=\; \sum_{\substack{0<l\le q\\ \gcd(l,q)=1}} e\left(\frac{lh}{q}\right)\pi(v;q,l)$$

and from the prime number theorem for arithmetic progressions (Theorem 2),

$$\left|\pi(v;q,l)-\frac{\mathrm{ls}(v)}{\phi(q)}\right| \;<\; \frac{n}{(\log n)^{100}},\qquad \gcd(l,q)=1.$$

Furthermore, by Lemma 1,

$$\mu(q) \;=\; c_q(h) \;=\; \sum_{0<l\le q,\ \gcd(l,q)=1} e\left(\frac{lh}{q}\right).$$

Hence, observing that $g(0,v) = \mathrm{ls}(v)$,

$$\begin{aligned}
\left|f\left(\frac{h}{q},v\right) - \frac{\mu(q)}{\phi(q)}\,g(0,v)\right| &< q + \left|\sum_{p\le v,\ p\nmid q} e\left(\frac{ph}{q}\right) - \frac{\mu(q)}{\phi(q)}\,\mathrm{ls}(v)\right| \\
&= q + \left|\sum_{0<l\le q,\ \gcd(l,q)=1} e\left(\frac{lh}{q}\right)\left(\pi(v;q,l) - \frac{\mathrm{ls}(v)}{\phi(q)}\right)\right| \\
&\le q + \frac{qn}{(\log n)^{100}} \;<\; \frac{2n}{(\log n)^{85}}.
\end{aligned}$$

Hence by Theorem 4 and Theorem 9,

$$\begin{aligned}
&\left|f\left(\frac{h}{q}+y,n\right) - \frac{\mu(q)}{\phi(q)}\,g(y,n)\right| \\
&= \left|e(ny)\left(f\left(\frac{h}{q},n\right) - \frac{\mu(q)}{\phi(q)}\,g(0,n)\right) - 2\pi i y\int_0^n e(vy)\left(f\left(\frac{h}{q},v\right) - \frac{\mu(q)}{\phi(q)}\,g(0,v)\right)dv\right| \\
&\le \left|f\left(\frac{h}{q},n\right) - \frac{\mu(q)}{\phi(q)}\,g(0,n)\right| + 2\pi x_0\int_0^n \left|f\left(\frac{h}{q},v\right) - \frac{\mu(q)}{\phi(q)}\,g(0,v)\right|dv \\
&\le \frac{2n(1+2\pi x_0 n)}{(\log n)^{85}} \;<\; \frac{14n}{(\log n)^{70}} \;<\; \frac{n}{(\log n)^{69}}.
\end{aligned}$$

□

Observe that for non-square-free q, $\mu(q) = 0$ and $f(x,n)$ is small when x is near h/q.

Theorem 11 (T59) *Suppose* $q < (\log n)^{15}$, $\gcd(h,q) = 1$ *and* $|y| \le x_0$. *Then*

$$\left|f\left(\frac{h}{q}+y,n\right)^3 - \frac{\mu(q)}{\phi(q)^3}\,g(y,n)^3\right| \;\le\; \frac{3n^3}{(\log n)^{69}}.$$

Proof This follows from Theorem 10 together with the trivial estimates $|f(x,n)| \le n$ and $|g(x,n)| \le n$. □

Substituting $x = h/q + y$ in the expression for $\mathcal{J}(h,q)$ gives

$$\mathcal{J}(h,q) \;=\; e\left(-\frac{nh}{q}\right)\int_{-x_0}^{x_0} f\left(\frac{h}{q}+y,n\right)^3 e(-ny)dy.$$

Putting

$$\mathcal{K} \;=\; \int_{-x_0}^{x_0} g(y,n)^3 e(-ny)dy,$$

we have by Theorem 11 and recalling that $x_0 = (\log n)^{15}/n$,

$$\left|\mathcal{J}(h,q) - \frac{\mu(q)}{\phi(q)^3}\,e\left(-\frac{nh}{q}\right)\mathcal{K}\right| \;\le\; \frac{6n^2}{(\log n)^{54}}, \tag{2}$$

provided that $\gcd(h,q)=1$ and $q \le (\log n)^{15}$.

Just as $r(n)$ can be expressed as an integral (1) involving $f(x,n)$, $\rho(n)$ has a similar formula using $g(x,n)$:

$$\rho(n) \;=\; \int_{-1/2}^{1/2} g(y,n)^3 e(-ny)dy.$$

Now for $w>2$ and $0<|y|\le 1/2$,

$$\left|\sum_{m=2}^{w} e(my)\right| \;=\; \left|\frac{e((w+1)y)-e(2y)}{e(y)-1}\right| \;\le\; \frac{1}{|\sin \pi y|} \;\le\; \frac{1}{2|y|}.$$

By Abel's lemma (summation by parts),

$$g(y,n) \;=\; \frac{1}{\log(n+1)}\sum_{m=2}^{n} e(my) - \sum_{k=2}^{n}\sum_{m=2}^{k} e(my)\left(\frac{1}{\log(k+1)}-\frac{1}{\log k}\right),$$

proved by reversing the order of summation. Hence

$$|g(y,n)| \;\le\; \frac{1}{2|y|\log(n+1)} + \frac{1}{2|y|}\sum_{k=2}^{n}\left(\frac{1}{\log k}-\frac{1}{\log(k+1)}\right) \;\le\; \frac{1}{|y|} \qquad \text{for } 0<|y|\le 1/2,$$

and therefore, recalling that $\mathcal{K}$ is like $\rho(n)$ but integrating over the shorter interval $[-x_0,x_0]$,

$$|\rho(n)-\mathcal{K}| \;\le\; \int_{-1/2}^{-x_0}\frac{dy}{|y|^3} + \int_{x_0}^{1/2}\frac{dy}{y^3} = 2\int_{x_0}^{1/2}\frac{dy}{y^3} \;<\; \frac{1}{x_0^2} \;=\; \frac{n^2}{(\log n)^{30}}.$$

From this and (2) we get a good estimate for a single major arc:

$$\left|\mathcal{J}(h,q) - \rho(n)\frac{\mu(q)}{\phi(q)^3}\, e\left(-\frac{nh}{q}\right)\right| \;\le\; \frac{6n^2}{(\log n)^{54}} + \frac{n^2}{\phi(q)^3(\log n)^{30}},$$

again provided that $\gcd(h,q)=1$ and $q\le(\log n)^{15}$. Summing over the major arcs and using the definition of Ramanujan's sum,

$$\left|\sum_{q\le(\log n)^{15}}\;\sum_{0<h\le q,\ \gcd(h,q)=1}\mathcal{J}(h,q) - \rho(n)\sum_{q\le(\log n)^{15}}\frac{\mu(q)}{\phi(q)^3}\,c_q(n)\right|$$

$$\le\; \frac{6n^2}{(\log n)^{24}} + \frac{n^2}{(\log n)^{30}}\sum_{q\le(\log n)^{15}}\frac{1}{\phi(q)^2} \;<\; \frac{7n^2}{(\log n)^{24}} \tag{3}$$

since $\sum_q \phi(q)^{-2}$ is bounded (recall that $\phi(n) > 0.3\,n^{0.9}$), and at last we have the estimate over the major arcs that we want.

Combining (3) with the minor arcs estimate (Theorem 8) gives

$$r(n) - \rho(n)\sum_{q\le(\log n)^{15}}\frac{\mu(q)}{\phi(q)^3}\,c_q(n) \;=\; O\left(\frac{n^2}{(\log n)^4}\right),$$

and it is easily shown that the same estimate holds when we take the sum to infinity. Recalling the definition of $S(n)$ from Theorem 3, we therefore have

$$r(n)-\rho(n)S(n) \;=\; O\left(\frac{n^2}{(\log n)^4}\right),$$

and the proof of Theorem 3 is complete.