Clifford Algebra and Computational Geometry

J.M. Selig

Faculty of Business, Computing and Information Management,

London South Bank University,

London SE1 0AA, UK

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1 Clifford Algebra

A Clifford algebra is an associative algebra (with a 1), that is a vector space with a product that distributes over the addition of vectors. As an algebra Clifford algebras are generated by a number of symbols, usually written e_1, e_2, ... and so forth. These symbols are subject to the following relations,

$$e_i e_j + e_j e_i = 2B_{ij}.$$

The juxtaposition of the e_is and e_js denote the Clifford multiplication and B_{ij} is a symmetric array of constants—a bilinear form. By taking a different set of generators, linear combinations of the original $e - i$s, it is always possible to reduce the matrix B_{ij} to diagonal form. So we may as well assume that B_{ij} has the form; $\mathrm{diag}(1, 1, \ldots, -1, -1, \ldots, 0, 0, \ldots)$. It is standard to denote such a Clifford algebra by $Cl(p, q, r)$ where p is the number of +1 eigenvalues, q the number of -1s and r the number of 0s. When there are no 0s we usually just write $Cl(p, q)$.

1.1 Example $Cl(0, 1)$

In this example there is just one generator and the single relation $e_1^2 = -1$. A typical element in the algebra has the form, $a + be_1$ where a and b are real coefficients. It is not hard to see that this is simple a fancy way of describing the complex numbers, $Cl(0, 1) = \mathbb{C}$.

1.2 Example $Cl(0, 2)$

Now there are two generators e_1 and e_2, both square to -1 and we have the further relation,

$$e_1 e_2 + e_2 e_1 = 0$$

that is $e_1 e_2 = -e_2 e_1$, the generators anti-commute. In this case a typical element of the algebra has the form,

$$a + be_1 + ce_2 + de_1 e_2.$$

Again a, b, c and d are real coefficients. Notice that as a vector space the algebra is 4-dimensional, in general it is not difficult to see that $Cl(p, q, r)$ will have dimension 2^{p+q+r}. The generators of the algebra as a vector space will be referred to as the basis elements of the algebra from now on.

Notice that we have the following products among the basis elements of the algebra,

$$
\begin{aligned}
(e_1)(e_2) &= e_1 e_2 \\
(e_1)(e_1 e_2) &= e_1^2 e_2 = -e_2 \\
(e_2)(e_1 e_2) &= -e_1 e_2^2 = e_1
\end{aligned}
$$

Moreover e_1 and e_2 anti-commute with the basis element $e_1 e_2$, since $(e_1 e_2)e_1 = -e_1^2 e_2 = e_2$ and $(e_1 e_2)e_2 = e_1 e_2^2 = -e_1$. This algebra can be identified with the algebra of Hamilton's quaternions, $Cl(0, 2) = \mathbb{H}$. In detail we put $\mathrm{i} = e_1$, $\mathrm{j} = e_2$ and $\mathrm{k} = e_1 e_2$, in this way Hamilton's famous relation becomes,

$$
\mathrm{ijk} = (e_1)(e_2)(e_1 e_2) = -e_1^2 e_2^2 = -1.
$$

Quaternions are important for representing rotations in 3D but in these note we will only be looking at problems in 2D. However, one of the advantages of using Clifford algebras is that generalising to higher dimensions is straightforward.

1.3 Example $Cl(0, 3)$

The basis for this algebra consists of the following eight elements,

$$
\begin{array}{ccc}
& 1, & \\
e_1, & e_2, & e_3, \\
e_1 e_2, & e_2 e_3, & e_3 e_1, \\
& e_1 e_2 e_3 &
\end{array}
$$

The grade of a basis element is defined to be the number of generators it contains. So this basis consists of one grade 0 element, three grade 1 elements three grade, 2 elements and a single grade 3 element.

Suppose we look at the elements of even grade, a typical even grade element has the form,

$$
a + b e_2 e_3 + c e_3 e_1 + d e_1 e_2
$$

In a product of basis elements the grade is not preserved in general, but since generators can only be simplified in pairs ($e_i^2 = \pm 1$) products of even grade basis elements will always result in even grade elements. Hence these elements form a sub-algebra, denoted $Cl^+(0, 3)$. In this case it is not hard to see that the even sub-algebra is once again isomorphic to the quaternions, $Cl^+(0, 3) = \mathbb{H} = Cl(0, 2)$.

In fact it is generally true that, $Cl^+(p, q+1, r) = Cl(p, q, r)$. To see this assume that $Cl(p, q, r)$ is generated by the elements $e_1, \ldots, e_n$ where $n = p + q + r$. The algebra $Cl(p, q+1, r)$ can be assumed to be generated by elements $a_1, \ldots, a_n$ and a_0, where $a_0^2 = -1$. Now we map the generators of $Cl(p, q, r)$ as follows,

$$
e_i \longmapsto a_i a_0
$$

and then extend the map to the whole of $Cl(p, q, r)$ by assuming that the map respects the algebra structure, in other words assume it is an algebra morphism. So for example, elements of $Cl^+(p, q + 1, r)$ of the form $a_i a_j$ with $i, j \neq 0$ are the image of elements $e_i e_j$ in $Cl(p, q, r)$. The two algebras clearly have the same dimension as vector spaces, hence all we need to do to prove the isomorphism is to check that the same relations hold in both algebras,

$$(a_i a_0)(a_j a_0) + (a_j a_0)(a_i a_0) = -(a_i a_j + a_j a_i)a_0^2 = (e_i e_j + e_j e_i).$$

2 The Exterior Product

On any Clifford algebra we can define another product called the exterior or Grassmann product. This is a derived product, since it is defined in terms of the Clifford product. The exterior product is defined for grade 1 elements and then extended to the rest of the algebra. So assume x is a grade 1 element, sometimes called a vector, and c is any arbitrary element of a Clifford algebra. The exterion product is defined by,

$$x \wedge c = \frac{1}{2}(xc + \alpha(c)x), \quad \text{and} \quad c \wedge x = \frac{1}{2}(cx + x\alpha(c)).$$

Here the map $\alpha()$ is known as the main involution of the algebra, it is defined on basis elements as,

$$\alpha(e_{i_1} e_{i_2} \cdots e_{i_k}) = (-1)^k e_{i_1} e_{i_2} \cdots e_{i_k}$$

The exterior product of generators is simple to evaluate,

$$e_i \wedge e_j = \frac{1}{2}(e_i e_j - e_j e_i) = \frac{1}{2}(e_i e_j + e_i e_j - 2B_{ij}) = e_i e_j - B_{ij}$$

Notice that if the generators are orthogonal, that is if B_{ij} is diagonal, then the exterior product will agree with the Clifford product except that the exterior product of any generator with itself will vanish. That is,

$$e_i \wedge e_j = e_i e_j, \quad \text{and} \quad e_i \wedge e_i = 0$$

3 Computational Geometry—Line Segments

Computational geometry studies algorithms to solve geometrical problems. A typical basic problem in Computational geometry is to find the convex hull of a given set of points. In any algorithm for this problem it is necessary to test whether or not a point lies on the right or left of a line determined by two other points. This is a basic predicate that must be computed. Often the study of these algorithms is only concerned with minimizing the number of calls to such a predicate. Here we look briefly at how the predicate itself could be evaluated.

Suppose we have three point in the plane,

$$p_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad p_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \quad \text{and} \quad p_3 = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}.$$

and we want to know whether or not p_3 is on the left of the line from p_1 to p_2. Recall that if u and v are two vectors in the plane then,

$$\det \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} = |u||v|\sin\theta,$$

where θ is the angle between the vectors measures from u to v with anti-clockwise positive. Suppose we write the points as Clifford algebra elements,

$$p_1 = x_1 e_1 + y_1 e_2, \quad p_2 = x_2 e_1 + y_2 e_2, \quad \text{and} \quad p_3 = x_3 e_1 + y_3 e_2$$

Then we have that,

$$
\begin{aligned}
(p_2 - p_1) \wedge (p_3 - p_1) &= ((x_2 - x_1)e_1 + (y_2 - y_1)e_2) \wedge ((x_3 - x_1)e_1 + (y_3 - y_1)e_2) \\
&= ((x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1))e_1 e_2 \\
&= \det \begin{pmatrix} (x_2 - x_1) & (x_3 - x_1) \\ (y_2 - y_1) & (y_3 - y_1) \end{pmatrix} e_1 e_2
\end{aligned}
$$

we can get rid of the $e_1 e_2$ by multiplying by $e_1 e_2$, remembering that $(e_1 e_2)(e_1 e_2) = -1$ we get that $\big((p_2 - p_1) \wedge (p_3 - p_1)\big)e_1 e_2$ is negative if p_3 is on the left of the line from p_1 to p_2, it is positive if p_3 is on the right of the line from p_1 to p_2 and it is zero if it lies on the line from p_1 to p_2. The cyclic symmetry of the expression can be seen more clearly if we expand the exterior products.

$$(p_2 - p_1) \wedge (p_3 - p_1) = p_1 \wedge p_2 + p_2 \wedge p_3 + p_3 \wedge p_1.$$

Remember here that $p_1 \wedge p_1 = 0$ and $p_1 \wedge p_2 = -p_2 \wedge p_1$ and so forth.

In fact we can write this a little more neatly in a bigger Clifford algebra. This time consider the Clifford algebra $Cl(0,3)$ generated by the elements e_0 and e_1, e_2. This time a point $p = (x, \ y)^T$, in the plane will be represented by an element,

$$\tilde{p} = e_0 + xe_1 + ye_2.$$

So that,

$$
\begin{aligned}
\tilde{p}_1 \wedge \tilde{p}_2 &= (e_0 + x_1 e_1 + y_1 e_2) \wedge (e_0 + x_2 e_1 + y_2 e_2) \\
&= (x_2 - x_1)e_0 e_1 + (y_2 y_1)e_0 e_2 + (x_1 y_2 - y_1 x_2)e_1 e_2.
\end{aligned}
$$

And hence,

$$\tilde{p}_1 \wedge \tilde{p}_2 \wedge \tilde{p}_3 = \big((x_1 y_2 - y_1 x_2) + (x_2 y_3 - y_2 x_3) + (x_3 y_1 - y_3 x_1)\big)e_0 e_1 e_2$$

which apart from the factor $e_0 e_1 e_2$ is clearly the same determinant as above. Another way to look at this is that the determinant we are trying to evaluate is,

$$\tilde{p}_1 \wedge \tilde{p}_2 \wedge \tilde{p}_3 = \det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} e_0 e_1 e_2$$

4

These computation can be applied to determining when line segments cross. Assume we have two line segments determined by their end points p_1, p_2 and p_3, p_4. For the line segments to cross we must have that p_3 and p_4 are on opposite sides of the line from p_1 to p_2. This can now easily be expressed as,

$$(\tilde{p}_1 \wedge \tilde{p}_2 \wedge \tilde{p}_3)(\tilde{p}_1 \wedge \tilde{p}_2 \wedge \tilde{p}_4) \leqslant 0.$$

It is easy to compute that $(e_0 e_1 e_2)^2 = 1$. This condition however, is not sufficient. We also need to ensure that p_1 and p_2 are on opposite sides of the line determined by p_3 and p_4,

$$(\tilde{p}_3 \wedge \tilde{p}_4 \wedge \tilde{p}_1)(\tilde{p}_3 \wedge \tilde{p}_4 \wedge \tilde{p}_2) \leqslant 0.$$

4 Computational Geometry—Circles

In this section we look at some computational problems involving circles. In Computational geometry these have direct applications to Delaunay triangulations, and hence to their duals; Voronoi diagrams.

Now any three points, p_1, p_2 and p_3 determine a circle. The equation of this circle is given by the determinant,

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x \\ y_1 & y_2 & y_3 & y \\ x_1^2 + y_1^2 & x_2^2 + y_2^2 & x_3^2 + y_3^2 & x^2 + y^2 \end{pmatrix} = 0.$$

To see this we can expand the determinant, assume that the circle has centre (c_x, c_y) and radius r and perform the following row operations on the determinant: take $2c_y$ times the third row $2c_x$ times the second row and $r^2 - c_x^2 - c_y^2$ times the first row from the fourth row. The result will be,

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x \\ y_1 & y_2 & y_2 & y \\ 0 & 0 & 0 & (x - c_x)^2 + (y - c_y)^2 - r^2 \end{pmatrix} =$$

$$\left((x - c_x)^2 + (y - c_y)^2 - r^2 \right) \det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}.$$

The zeros in the last row occur because we are assuming that the three points lie on the circle. As we saw above the 3×3 determinant on the right is non-zero if the three points are not colinear.

Next suppose we substitute a fourth point p_4 into the determinant, to give the expression,

$$\Delta(p_1, p_2, p_3, p_4) = \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ x_1^2 + y_1^2 & x_2^2 + y_2^2 & x_3^2 + y_3^2 & x_4^2 + y_4^2 \end{pmatrix}.$$

This expression will be negative if p_4 is inside the circle defined by p_1, p_2, p_3 provided the points p_1, p_2, p_3 are ordered anti-clockwise around the circle. This is simple to see now because clearly,

$$\Delta(p_1, p_2, p_3, p_4) = \left((x_4 - c_x)^2 + (y_4 - c_y)^2 - r^2\right) \det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$$

The factor $\left((x_4 - c_x)^2 + (y_4 - c_y)^2 - r^2\right)$ is clearly negative if p_4 lies inside the circle. As we saw above, the 3×3 determinant is positive if p_3 is on the left of the line from p_1 to p_2 and this is equivalent to the points being ordered anti-clockwise around the circle.

Again we can write these expressions as operations in a Clifford algebra. On possible solution would be to use the algebra $Cl(0, 4)$ and write the points as,

$$\hat{p} = e_0 + xe_1 + ye_2 + (x^2 + y^2)e_\infty$$

where the extra generators are e_0 which represents the point at the origin and e_∞ which represents a single point at infinity. The predicate is then given by,

$$\hat{p}_1 \wedge \hat{p}_2 \wedge \hat{p}_3 \wedge \hat{p}_4 = \Delta(p_1, p_2, p_3, p_4)e_0 e_1 e_2 e_\infty.$$

However, there are other representations which may be more useful.

5   Conclusions

The above represent the first steps in an attempt to use Clifford algebras in Computational geometry. So elementary operations in Computational geometry have been represented as Clifford algebra operations. Essentially determinants have been represented using exterior algebra. However, there is much more to Clifford algebra, in particular Clifford algebras contain important symmetry groups and their representations. So the hope is that this extra structure can be of use in Computational geometry.