Screw Systems and Their Classification

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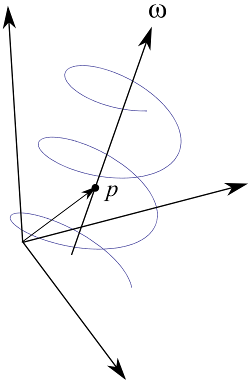
FW12: Geometric Representations: The Roles of Screw Theory, Lie algebra, & Geometric Algebra
ICRA London
2 June 2023

Introduction

- Screw systems are important and useful in many areas of Robotics and other areas.
- Classical subject but largely neglected in most contemporary work. R.S. Ball (1840–1913).
- A *twist* is an element of the Lie algebra $se(3)$, the Lie algebra to the groups of rigid-body displacement $SE(3)$.
- Classification by Hunt in 1970s but no proofs, Gibson and Hunt 1990s put it on a sound Mathematical footing.

What are Screws?



A Screw is an element of the projective space formed from $se(3)$. An equivalence class of twists, equivalent up to multiplication by a non-zero constant. Rays through the zero vector in $se(3)$. Line with a pitch.

Can write a twist (or screw) as a 6-component vector, partitioned as two 3-vectors,

$$\mathbf{s} = \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix}$$

Here $\boldsymbol{\omega}$ can be thought of as the angular velocity of of the rigid-body and $\mathbf{v}$ is the linear velocity of a point on the body instantaneously coincident with the origin.

What are Screw systems?

A *screw system* is a vector subspaces of the Lie algebra to the group of rigid-body displacements.

Given n linearly independent twists, $\mathbf{s}_1$, $\mathbf{s}_2, \ldots, \mathbf{s}_n$, their span gives a screw system. That, is the twists in the system are given by linear combinations,

$$\mathbf{s} = \lambda_1 \mathbf{s}_1 + \lambda_2 \mathbf{s}_2 + \cdots + \lambda_n \mathbf{s}_n$$

where the λ_is are real parameters.

The linear span of n basis elements gives an n-dimensional vector subspace, usually called an n-system of screws, (should really be twists!).

Where to find screw systems I

Jacobians for serial robots

The angular and linear velocity of the end-effector of a robot is given by a twist,

$$\mathbf{s} = J\dot{\boldsymbol{\theta}},$$

where J is the Jacobian matrix of the robot and $\dot{\boldsymbol{\theta}}$ is the vector of joint-rates.

The columns of the Jacobian matrix can be shown to be the twists associated with the joints of the robot, so if the robot has $n < 6$ joints, the twist at the end-effector can be written,

$$\mathbf{s} = \dot{\theta}_1\mathbf{s}_1 + \dot{\theta}_2\mathbf{s}_2 + \cdots + \dot{\theta}_n\mathbf{s}_n.$$

So, as the joint rates $\dot{\theta}_i$ change the end-effector twist velocity can only range over the screw system generated by $\mathbf{s}_1, \ldots, \mathbf{s}_n$.

Wrenches

Duals of twists can represent forces and torques. These can be written as 6-vectors partitioned as,

$$\mathcal{W} = \begin{pmatrix} \boldsymbol{\tau} \\ \mathbf{F} \end{pmatrix},$$

and called wrenches. Here, $\mathbf{F}$ is the force acting on the body and $\boldsymbol{\tau}$ is the moment acting on the body.

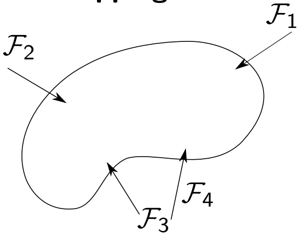
If a rigid body is moving with velocity twist $\mathbf{s}$ and is acted on by a wrench $\mathcal{W}$, then the power being expended is given by,

$$\mathcal{W}^T\mathbf{s} = \boldsymbol{\tau} \cdot \boldsymbol{\omega} + \mathbf{F} \cdot \mathbf{v} = \text{Power}$$

Classically, no difference between twists and wrenches. Certainly, classification of screw or wrench systems the same.

Where to find screw systems II

Gripping



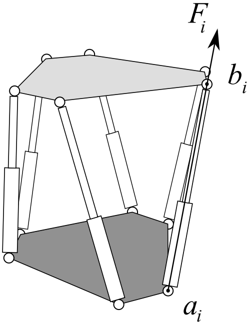
Fingers produce forces along normal lines to the surface of the object to be grasped. For n fingers we get n unit wrenches $\mathcal{F}_i$ so the total wrench on the object is given by multiplying the by the intensities λ_i of the finger forces,

$$\mathcal{W} = \lambda_1\mathcal{F}_1 + \lambda_2\mathcal{F}_2 + \cdots \lambda_n\mathcal{F}_n$$

Body in equilibrium if and only if the total wrench vanishes. So, only wrenches that can be balanced, e.g. gravity, must lie in the wrench system generated by the finger wrenches.

Where to find screw systems III

Parallel Robot Jacobian



The Jacobian J, is the matrix satisfying,

$$\dot{\mathbf{L}} = J\mathbf{s}$$

where $\mathbf{s}$ is the velocity twist of the platform and $\dot{\mathbf{L}} = (\dot{L}_1, \dot{L}_2, \ldots, \dot{L}_n)^T$, is the vector of joint velocities.

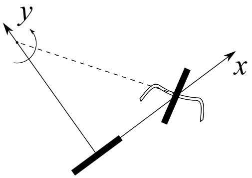
The rows of this Jacobian are the wrenches,

$$\mathcal{W}_i^T = \frac{1}{L_i}\big((\mathbf{a}_i \times \mathbf{b}_i)^T, (\mathbf{b}_i - \mathbf{a}_i)^T\big), \quad i = 1, \ldots, n$$

Only wrenches in the system defined by the rows of the Jacobian can be applied to the platform.

Where to find screw systems IV

Mobile Robots.



The propulsion systems and controls for many vehicles are fixed on the body of the vehicle. So, in the body-fixed frame of the vehicle the forces and torques driving the vehicle are fixed. For the bicycle or car, the vehicle can only turn about a point located on the line of the rear axle. Pure translation can be thought of as rotation about a point at infinity.

The kinematic equation for the vehicle can be written as,

$$\frac{dG}{dt} = GS$$

Here, G is the 4×4 matrix representing the displacement of the vehicle, ...

Where to find screw systems IV - Continued

and S is velocity twist written as a 4×4 matrix,

$$S = \begin{pmatrix} \Omega & \mathbf{v} \\ 0 & 0 \end{pmatrix}$$

with Ω the 3×3 anti-symmetric matrix corresponding to $\boldsymbol{\omega}$. Possible propulsion intensities and control actions limit S to particular screw system.

Other examples, aeroplanes and submarines, also needle steering, Frenet-Serret and Bishop motions. Last two important for motion planning.

What is a classification?

In Maths “classification” has a precise meaning.

Need an equivalence relation. Here, two screw systems are considered equivalent if there is a rigid-body displacement that transforms one system into the other, or if there is a change in basis which transforms the basis of one into the basis of the other.

The classification problem is to find all possible equivalence classes. The list of classes must be totally exhaustive and mutually exclusive.

Group Action

The adjoint action of $SE(3)$ on its Lie algebra is given by,

$$\mathbf{s} = \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix} \longmapsto \begin{pmatrix} R & 0 \\ TR & R \end{pmatrix} \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix} = \mathrm{Ad}(G)\mathbf{s}$$

Here, R is the usual 3×3 rotation matrix of the displacement and T is the 3×3 anti-symmetric matrix corresponding to the translation vector $\mathbf{t}$.

The co-adjoint action on wrenches is given by,

$$\mathcal{W} = \begin{pmatrix} \boldsymbol{\tau} \\ \mathbf{F} \end{pmatrix} \longmapsto \begin{pmatrix} R & TR \\ 0 & R \end{pmatrix} \begin{pmatrix} \boldsymbol{\tau} \\ \mathbf{F} \end{pmatrix} = \mathrm{Ad}(G)^{-T}\mathcal{W}$$

This is the inverse-transpose of the adjoint representation — so that the power remains a scalar.

Inner Products

Only two bi-invariant quadratic forms on twists. General left-invariant quadratic form can be written as,

$$q(\mathbf{s}_1, \mathbf{s}_2) = \mathbf{s}_1^T Q \mathbf{s}_2$$

where Q is a 6×6 symmetric matrix. For right invariance we must have,

$$\mathrm{Ad}(G)^T Q \, \mathrm{Ad}(G) = Q, \quad \text{for all} \quad G \in SE(3)$$

Solutions are,

$$Q_0 = \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix} \quad \text{and} \quad Q_\infty = \begin{pmatrix} 2I_3 & 0 \\ 0 & 0 \end{pmatrix}$$

or any linear combination $\alpha Q_0 + \beta Q_\infty$.
Q_0 — Klein form/Reciprocal product. Q_∞ — Killing form.

Pitch

The *pitch* of a screw is an invariant of the screw $\mathbf{s}$, given by,

$$p = \frac{\mathbf{s}^T Q_0 \mathbf{s}}{\mathbf{s}^T Q_\infty \mathbf{s}} = \frac{\boldsymbol{\omega} \cdot \mathbf{v}}{\boldsymbol{\omega} \cdot \boldsymbol{\omega}}$$

If $\mathbf{s}^T Q_0 \mathbf{s} = 0$ the pitch is zero, and the screw is a line. This happens when $\boldsymbol{\omega}$ and $\mathbf{v}$ are perpendicular. So, $\mathbf{v} = \mathbf{p} \times \boldsymbol{\omega}$ is the moment of the line, where $\mathbf{p}$ is any point on the line.

If $\mathbf{s}^T Q_\infty \mathbf{s} = 0$, we must have $\boldsymbol{\omega} = \mathbf{0}$, so the screw has no angular velocity and is an instantaneous pure translation. Such screws are said to have infinite pitch.

1-Systems

1-systems of screws, that is single screws are completely classified by their pitch. Every screw has a pitch, possibly infinite, no rigid body displacement can change the pitch of a screw. All possible values of the pitch, even infinity, are possible.

The equivalence classes of 1-systems form a continuous family parameterised by the pitch p.

Reciprocal Systems

If $\mathbf{s}_1^T Q_0 \mathbf{s}_2 = 0$ then the screws $\mathbf{s}_1$ and $\mathbf{s}_2$ are said to be reciprocal. Clearly, if a screw $\mathbf{s}_3$ is reciprocal to both $\mathbf{s}_1$ and $\mathbf{s}_2$ then it is reciprocal to any linear combination of $\mathbf{s}_1$ and $\mathbf{s}_2$,

$$\mathbf{s}_1^T Q_0 \mathbf{s}_3 = \mathbf{s}_2^T Q_0 \mathbf{s}_3 = 0 \quad \Rightarrow \quad (\lambda_1 \mathbf{s}_1 + \lambda_2 \mathbf{s}_2)^T Q_0 \mathbf{s}_3 = 0,$$

for all values of λ_1 and λ_2.

Given an n-system of screws, $\mathrm{S} = \mathrm{Span}(\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n)$, the set of all screws reciprocal to every screw in S form a screw system called the reciprocal screws system to S,

$$\overline{\mathrm{S}} = \{\mathbf{s} \in se(3) \, : \, \mathbf{s}^T Q_0 \mathbf{s}_1 = \mathbf{s}^T Q_0 \mathbf{s}_2 = \cdots = \mathbf{s}^T Q_0 \mathbf{s}_n = 0\}$$

This is a $(6-n)$-system.

Reciprocal Systems - Continued

Each distinct n-system has a distinct $(6-n)$-reciprocal system. Further the reciprocal system to a reciprocal system, (taking the reciprocal system twice) yeilds the original system.

This means that we only have to classify $1, 2$ and 3 systems of screws. 4-systems are classified by their reciprocal 2-systems, and 5-systems are classified by their reciprocal 1-systems. That is 5-systems of screws have a pitch, given by the pitch of their reciprocal 1-system.

The pitch of a 5-system can be found directly as follows, suppose the generators of the 5-system are $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_5$, let J be the 6×5 matrix with these screws as columns. Now evaluate the 5×5 determinant,

$$\det\left(J^T(\alpha Q_0 + \beta Q_\infty)J\right) = K\alpha^5 + L\alpha^4\beta$$

The pitch of the 5-system is given by the ratio of the coefficients,

$$p = -K/L$$

Sketch of the Classification

To classify 2 and 3-systems: The number of infinite pitch screws in the system is clearly an invariant.

- A-systems have no infinite pitch screws
- B-systems have a single infinite pitch screw
- C-systems have a line of infinite pitch screws
- D-systems contain all infinite pitch screws

Only one 3-system contains all infinite pitch screws.

If the rest of the screws in the system have the same pitch the system is referred to as a II-system, otherwise it is a I-system.

Example I

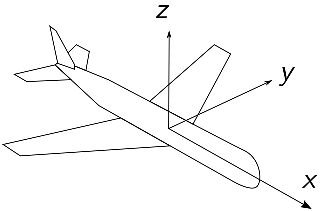
So, for example a 2-system classified as IIB ($p = 0$) consists of pitch 0 screws and a single infinite pitch screw.

This is the screw system mentioned in connection with cars and bicycles. It can be generated by the basis screws,

$$\mathbf{s}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{s}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Notice $\mathbf{s}_2$ has infinite pitch and any linear combination of $\mathbf{s}_1$ and $\mathbf{s}_2$ has pitch 0. The system consists of all the lines parallel to the z-axis passing through points on the y-axis plus the infinite pitch screw directed parallel to the x-axis.

Example II



Aeroplanes can roll, pitch, yaw and fly forwards. The generators of this 4-system in the body-fixed frame of the aeroplane can be written,

$$\mathbf{s}_1 = \begin{pmatrix} \mathbf{i} \\ \mathbf{0} \end{pmatrix}, \mathbf{s}_2 = \begin{pmatrix} \mathbf{j} \\ \mathbf{0} \end{pmatrix}, \mathbf{s}_3 = \begin{pmatrix} \mathbf{k} \\ \mathbf{0} \end{pmatrix}, \mathbf{s}_4 = \begin{pmatrix} \mathbf{0} \\ \mathbf{i} \end{pmatrix}$$

A basis for the screws reciprocal to these 4 is given by,

$$\mathbf{s}_5 = \begin{pmatrix} \mathbf{j} \\ \mathbf{0} \end{pmatrix}, \mathbf{s}_6 = \begin{pmatrix} \mathbf{k} \\ \mathbf{0} \end{pmatrix}$$

These are lines (pitch 0 screws) through the origin, so form a IIA ($p = 0$) 2-system. The 4-system controlling the aeroplane is thus a $\overline{\text{IIA}}\,(p = 0)$ system.

Conclusions

- Sketch gives discrete types, most of these contain one or two parameter families of equivalence classes, for example different pitch II-systems and pitches of principal screws in I-systems.
- Special types of screw systems, subalgebras and Lie triple systems. Usually very important in applications.
- Lots of geometry associated with screw systems, in particular line geometry. The lines in a 5-system form a linear line complex, the lines in a 4-system form linear line congruences. The axes of the screws in a 2-system form a ruled surface.