Some Remarks on the Rigid-Body Dynamics of Robots

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Introduction

For rigid-body dynamics, students are usually taught to separate rotational and translations motion. Not the best way for systems with several rigid bodies.

Look at formalism (screw theory) that doesn't do this. Makes setting-up equations of motion simpler and allows reasoning about design.

Also in robotics, many moving coordinate frames used. Simpler to use a single fixed frame.

Wrenches

Wrenches are vectors combining the force and moment acting on a rigid body. Can write wrenches as 6-component vectors or in partitioned form as

$$\mathcal{W} = \begin{pmatrix} \vec{\tau} \\ \vec{F} \end{pmatrix} = \begin{pmatrix} \tau_x \\ \tau_y \\ \tau_z \\ F_x \\ F_y \\ F_z \end{pmatrix}$$

Notice, components have different units. Have to be careful not to mix units.

Group Action

Suppose we have a wrench acting on a body, say a finger of a robot gripper. Can describe how the wrench changes if the system is subject to a rigid-body displacement. Any rigid displacement is composed of a rotation and a translation. Effect on a wrench $\mathcal{W}$ given by,

$$\mathcal{W} \longrightarrow \mathrm{Ad}^*(g)\mathcal{W} = \begin{pmatrix} R & TR \\ 0 & R \end{pmatrix} \begin{pmatrix} \vec{\tau} \\ \vec{F} \end{pmatrix}$$

where R is the 3×3 rotation matrix of the displacement g and T is the translation written as a 3×3 anti-symmetric matrix.

$$T = \begin{pmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{pmatrix}, \quad \text{so } T\vec{x} = \vec{t} \times \vec{x} \text{ for any } \vec{x}$$

Group of all rigid-body displacements denoted $SE(3)$.

Metrics

There is no positive definite bi-invariant metric on wrenches — recall, can't mix units.

But there are two indefinite bi-invariant quadratic forms on wrenches,

$$\mathcal{W}^T Q_\infty^* \mathcal{W} \quad \text{and} \quad \mathcal{W}^T Q_0^* \mathcal{W}$$

where,

$$Q_\infty^* = \begin{pmatrix} 0 & 0 \\ 0 & 2I_3 \end{pmatrix} \qquad Q_0^* \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}$$

where I_3 is the 3×3 identity matrix.

Any linear combination of these forms is bi-invariant. A rigid displacement gives,

$$\mathrm{Ad}^*(g)^T \left(\alpha Q_\infty^* + \beta Q_0^*\right) \mathrm{Ad}^*(g) = \left(\alpha Q_\infty^* + \beta Q_0^*\right)$$

for arbitrary α and β and all $g \in SE(3)$

Pitch of a Wrench

Wrenches have a line of action and a pitch. The pitch of a wrench is invariant under rigid displacements,

$$\text{pitch} = \frac{\mathcal{W}^T Q_0^* \mathcal{W}}{\mathcal{W}^T Q_\infty^* \mathcal{W}} = \frac{\vec{F} \cdot \vec{\tau}}{\vec{F} \cdot \vec{F}}$$

Pure force has pitch 0, pure couple has $\vec{F} = \vec{0}$, infinite pitch (by convention). General wrench can be written,

$$\mathcal{W} = \begin{pmatrix} \vec{r} \times \vec{F} + p\vec{F} \\ \vec{F} \end{pmatrix}$$

where p is the pitch and $\vec{r}$ is any point on the line of action on the wrench.

Twists

Twists are dual to wrenches. Elements of the Lie algebra to $SE(3)$. Used to represent general velocities - angular and linear. Also "small" displacements, pose errors.

General twist

$$\mathbf{s} = \begin{pmatrix} \vec{\omega} \\ \vec{v} \end{pmatrix} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \\ v_x \\ v_y \\ v_z \end{pmatrix}$$

where $\vec{\omega}$ is the angular velocity of the body and $\vec{v}$ is the velocity of a point on the body instantaneously coincident with the origin.

Power

Duality means we can pair twists and wrenches to produce a scalar - a quantity independent of coordinates. If a wrench $\mathcal{W}$, acts on a rigid body and produces a twist $\mathbf{s}$, the scalar is the power expended

$$\mathcal{W}^T \mathbf{s} = \begin{pmatrix} \vec{\tau}^T & \vec{F}^T \end{pmatrix} \begin{pmatrix} \vec{\omega} \\ \vec{v} \end{pmatrix} = \vec{\tau} \cdot \vec{\omega} + \vec{F} \cdot \vec{v} = \text{power}$$

If $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_4$, $\mathcal{W}_5$, $\mathcal{W}_6$ form a basis for the 6-dimensional vector space of wrenches then we can always find a dual basis of twists $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{s}_3$, $\mathbf{s}_4$, $\mathbf{s}_5$, $\mathbf{s}_6$. That is, we have

$$\mathcal{W}_i^T \mathbf{s}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Can use variant of Gram–Schmidt process but often easy to find by inspection.

Group Action

Adjoint action of the group $SE(3)$ on its Lie algebra.

$$\mathbf{s} \longmapsto \mathrm{Ad}(g)\mathbf{s} = \begin{pmatrix} R & 0 \\ TR & R \end{pmatrix} \begin{pmatrix} \vec{\omega} \\ \vec{v} \end{pmatrix}$$

as with wrenches R is the rotation matrix of the displacement and T gives the translation. Note that, action on wrenches given by inverse-transpose of this

$$Ad^*(g) = Ad^{-T}(g) \qquad \text{for all} \quad g \in SE(3)$$

Twists also have a line of action and a pitch

$$\text{pitch} = \frac{\mathbf{s}^T Q_0 \mathbf{s}}{\mathbf{s}^T Q_\infty \mathbf{s}} = \frac{\vec{\omega} \cdot \vec{v}}{\vec{\omega} \cdot \vec{\omega}} \quad \text{where } Q_0 = \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}, \; Q_\infty = \begin{pmatrix} 2I_3 & 0 \\ 0 & 0 \end{pmatrix}$$

Momentum

The angular momentum and linear momentum of a rigid body can be combined into a 6-vector

$$\mathcal{M} = \begin{pmatrix} \vec{\jmath} \\ \vec{p} \end{pmatrix}$$

where $\vec{\jmath}$ is the angular momentum of the body and $\vec{p}$ its linear momentum. We also have

$$\vec{\jmath} = \mathrm{I}\,\vec{\omega} + m(\vec{c} \times \vec{v}) \quad \text{and} \quad \vec{p} = m\vec{v} + m(\vec{\omega} \times \vec{c})$$

where I is the usual 3×3 inertia matrix of the body, m its mass and $\vec{c}$ the position vector of its centre of mass.

Note, this 6-momentum transforms with $\mathrm{Ad}^*(g)$, like a wrench.

Inertia

The previous relations for the momentum allow us to define the 6×6 inertia matrix of the body

$$N = \begin{pmatrix} \mathrm{I} & mC \\ mC^T & mI_3 \end{pmatrix}$$

where C is the 3×3 matrix corresponding to $\vec{c}$, that is $C\vec{x} = \vec{c} \times \vec{x}$ for any vector $\vec{x}$.

So the equations for the momenta can be written

$$\mathcal{M} = N\mathbf{s}$$

Equations of Motion for a Rigid Body

According to Newton's laws the rate of change of momentum is equal to the applied force. Combining this with Euler's laws for angular motion gives,

$$\frac{d}{dt}\mathcal{M} = \frac{d}{dt}(N\mathbf{s}) = \mathcal{W}$$

where $\mathcal{W}$ is the applied wrench.

This only holds in an inertial frame, like the global frame.

Action of $SE(3)$ on the Inertia

Notice that, as the body moves the inertia changes.

How does N change under a rigid displacement? Since the momentum $\mathcal{M} = N\mathbf{s}$, transforms a wrench we have

$$N\mathbf{s} \longmapsto \mathrm{Ad}^*(g)N\mathbf{s}$$

but $\mathbf{s}$ transforms according to the adjoint representation, so to maintain the above relation we must have that

$$N \longmapsto \mathrm{Ad}^*(g)N\,\mathrm{Ad}^{-1}(g)$$

So, as the body moves according to some sequence of displacements $g(t)$, the inertia changes as

$$N(t) = \mathrm{Ad}^*\left(g(t)\right)N_0\,\mathrm{Ad}^{-1}\left(g(t)\right)$$

Where N_0 is the inertia matrix at time $t = 0$.

Velocity Twist

Given a rigid body motion $g(t)$, the instantaneous velocity twist of the body is given by,

$$\mathrm{ad}(\mathbf{s}) = \frac{d}{dt}\left(\mathrm{Ad}\left(g(t)\right)\right)\mathrm{Ad}^{-1}\left(g(t)\right)$$

(representation doesn't really matter much.) Here, if

$$\mathbf{s} = \begin{pmatrix} \vec{\omega} \\ \vec{v} \end{pmatrix}$$

then

$$\mathrm{ad}(\mathbf{s}) = \begin{pmatrix} \Omega & 0 \\ V & \Omega \end{pmatrix}$$

with Ω and v the 3×3 matrices for $\vec{\omega}$ and $\vec{v}$. Note, $\mathrm{ad}()$ different from $\mathrm{Ad}()$.

Lie Brackets

Above gives algebraic operation on twists,

$$[\mathbf{s}_1, \mathbf{s}_2] = \mathrm{ad}(\mathbf{s_1})\mathbf{s}_2 = \begin{pmatrix} \vec{\omega}_1 \times \vec{\omega}_2 \\ \vec{v}_1 \times \vec{\omega}_2 + \vec{\omega}_1 \times \vec{v}_2 \end{pmatrix}$$

Gives twists the structure of a Lie algebra.

Anti-symmetric,

$$[\mathbf{s}_1, \mathbf{s}_2] = -[\mathbf{s}_2, \mathbf{s}_1]$$

so $[\mathbf{s}, \mathbf{s}] = 0$.

Non-associative, but satisfies Jacobi identity

$$\big[\mathbf{s}_1, [\mathbf{s}_2, \mathbf{s}_3]\big] + \big[\mathbf{s}_2, [\mathbf{s}_3, \mathbf{s}_1]\big] + \big[\mathbf{s}_3, [\mathbf{s}_1, \mathbf{s}_2]\big] = 0$$

More Derivatives

Differentiating $\mathrm{Ad}\left(g(t)\right)\mathrm{Ad}^{-1}\left(g(t)\right) = I_6$ can show that

$$\frac{d}{dt}\left(\mathrm{Ad}^{-1}\left(g(t)\right)\right) =$$

$$-\mathrm{Ad}^{-1}\left(g(t)\right)\frac{d}{dt}\left(\mathrm{Ad}\left(g(t)\right)\right)\mathrm{Ad}^{-1}\left(g(t)\right) =$$

$$-\mathrm{Ad}^{-1}\left(g(t)\right)\mathrm{ad}(\mathbf{s})$$

Transposing gives

$$\frac{d}{dt}\left(\mathrm{Ad}^{*}\left(g(t)\right)\right) = -\mathrm{ad}^{T}(\mathbf{s})\,\mathrm{Ad}^{*}\left(g(t)\right)$$

Equations of Motion Again

Recall, equations of motion for a rigid body are

$$\frac{d}{dt}(N\mathbf{s}) = \mathcal{W}$$

Now expand the derivatives

$$\frac{d}{dt}(N\mathbf{s}) = N\frac{d}{dt}\mathbf{s} + \frac{d}{dt}(N)\mathbf{s}$$

Now, $N = \mathrm{Ad}^*\left(g(t)\right) N_0 \,\mathrm{Ad}^{-1}\left(g(t)\right)$ with N_0 the inertia at $t = 0$, so

$$\frac{d}{dt}(N) = -\,\mathrm{ad}^T(\mathbf{s})N - N\,\mathrm{ad}(\mathbf{s})$$

Equations of Motion continued

So

$$\frac{d}{dt}(N)\mathbf{s} = -\operatorname{ad}^T(\mathbf{s})N\mathbf{s}$$

Recall, $\operatorname{ad}(\mathbf{s})\mathbf{s} = 0$. This is the co-adjoint action of a twist on a wrench. Arnol'd notation

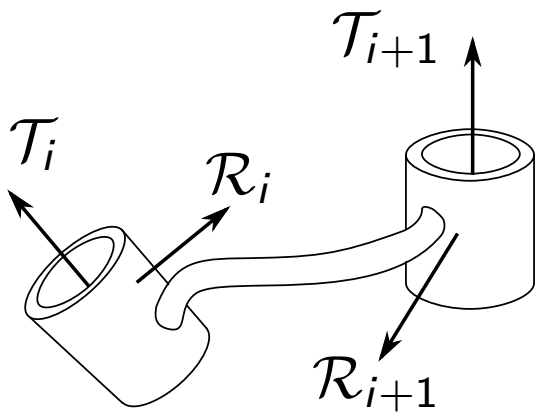
$$\{\mathbf{s}, \mathcal{W}\} = -\operatorname{ad}^T(\mathbf{s})\mathcal{W} = \begin{pmatrix} \vec{\omega} \times \vec{\tau} + \vec{v} \times \vec{F} \\ \vec{\omega} \times \vec{F} \end{pmatrix}$$

Equations of motion can now be written

$$N\dot{\mathbf{s}} + \{\mathbf{s}, N\mathbf{s}\} = \mathcal{W}$$

Single Robot Link

At last! Consider the ith link in a serial robot. Ignore gravity for the moment.



As above equation of motion, in the global fixed frame simply,

$$N_i\dot{\mathbf{s}}_i + \{\mathbf{s}_i,\, N_i\mathbf{s}_i\} = \mathcal{W}_i$$

where the subscript i refers to the ith link. But now we can say a little more about the applied wrench—it is composed of the joint torques and the reaction wrenches.

$$N_i\dot{\mathbf{s}}_i + \{\mathbf{s}_i,\, N_i\mathbf{s}_i\} = \mathcal{T}_i + \mathcal{R}_i - \mathcal{T}_{i+1} - \mathcal{R}_{i+1}$$

The Last Link

The equation for the final link, usually link 6, will be a little different,

$$N_6\dot{\mathbf{s}}_6 + \{\mathbf{s}_6,\, N_6\mathbf{s}_6\} = \mathcal{T}_6 + \mathcal{R}_6$$

Could include effect of payload by modifying N_6.

Reaction wrench can be eliminated by pairing with the joint twist of the final joint $\mathbf{z}_6$,

$$\mathbf{z}_6^T N_6\dot{\mathbf{s}}_6 + \mathbf{z}_6^T \{\mathbf{s}_6,\, N_6\mathbf{s}_6\} = \tau_6$$

Joint Twists

The joints of the robot are represented by twist $\mathbf{z}_i$. For revolute joints these are pitch zero twists with line given by the axis of the joint. So

$$\mathbf{z}_i = \begin{pmatrix} \vec{\omega}_i \\ \vec{r}_i \times \vec{\omega}_i \end{pmatrix}$$

with $\vec{\omega}_i$ the unit vector in the direction of the joint axis and $\vec{r}_i$ any point on the axis of the ith joint.

Reaction wrenches can do no work on the joint twist so $\mathbf{z}_6^T \mathcal{R}_6 = \mathcal{R}_6^T \mathbf{z}_6 = 0$ (no power).

The motor torque is about the joint's axis so that

$$\mathcal{T}_i = \begin{pmatrix} \tau_i \vec{\omega}_i \\ \vec{0} \end{pmatrix}$$

Hence, $\mathbf{z}_6^T \mathcal{T}_6 = \mathcal{T}_6^T \mathbf{z}_6 = \tau_6$ the torque provided by the motor at joint i.

The Other Links

writing all the equations for all the links gives,

$$
\begin{aligned}
N_1\dot{\mathbf{s}}_1 + \left\{\mathbf{s}_1,\, N_i\mathbf{s}_1\right\} &= \mathcal{T}_1 + \mathcal{R}_1 - \mathcal{T}_2 - \mathcal{R}_2 \\
N_2\dot{\mathbf{s}}_2 + \left\{\mathbf{s}_2,\, N_2\mathbf{s}_2\right\} &= \mathcal{T}_2 + \mathcal{R}_2 - \mathcal{T}_3 - \mathcal{R}_3 \\
N_3\dot{\mathbf{s}}_3 + \left\{\mathbf{s}_3,\, N_3\mathbf{s}_3\right\} &= \mathcal{T}_3 + \mathcal{R}_3 - \mathcal{T}_4 - \mathcal{R}_4 \\
N_4\dot{\mathbf{s}}_4 + \left\{\mathbf{s}_4,\, N_4\mathbf{s}_4\right\} &= \mathcal{T}_4 + \mathcal{R}_4 - \mathcal{T}_5 - \mathcal{R}_5 \\
N_5\dot{\mathbf{s}}_5 + \left\{\mathbf{s}_5,\, N_5\mathbf{s}_5\right\} &= \mathcal{T}_5 + \mathcal{R}_5 - \mathcal{T}_6 - \mathcal{R}_6 \\
N_6\dot{\mathbf{s}}_6 + \left\{\mathbf{s}_6,\, N_6\mathbf{s}_6\right\} &= \mathcal{T}_6 + \mathcal{R}_6
\end{aligned}
$$

Can add cumulatively from the bottom to eliminate torques and reactions from higher links,

$$
\sum_{i=j}^{6}\left(N_6\dot{\mathbf{s}}_6 + \left\{\mathbf{s}_6,\, N_6\mathbf{s}_6\right\}\right) = \mathcal{T}_j + \mathcal{R}_j, \qquad j = 1, 2, \ldots 6
$$

Equations of Motion for a Serial Robot

Now pair with joint twist to get.

$$\sum_{i=j}^{6} \mathbf{z}_j^T \left(N_i \dot{\mathbf{s}}_i + \{\mathbf{s}_i,\, N_i \mathbf{s}_i\} \right) = \tau_j, \qquad j = 1, 2, \ldots 6$$

This can be rearranged to give

$$\sum_{i=j}^{6} \left(\mathbf{z}_j^T N_i \dot{\mathbf{s}}_i + \mathbf{s}_i^T N_i [\mathbf{z}_j,\, \mathbf{s}_i] \right) = \tau_j, \qquad j = 1, 2, \ldots 6$$

Notice 6 scalar equations.

Homework: show that $\mathbf{s}_1^T \{\mathbf{s}_2,\, \mathcal{W}\} = \mathcal{W}^T [\mathbf{s}_1,\, \mathbf{s}_2]$

Including Gravity

How do these equations change if we include gravity?
The link equations gain an extra term each,

$$
\begin{aligned}
N_1\dot{\mathbf{s}}_1 + \left\{\mathbf{s}_1,\, N_i\mathbf{s}_1\right\} &= \mathcal{G}_1 + \mathcal{T}_1 + \mathcal{R}_1 - \mathcal{T}_2 - \mathcal{R}_2 \\
N_2\dot{\mathbf{s}}_2 + \left\{\mathbf{s}_2,\, N_2\mathbf{s}_2\right\} &= \mathcal{G}_2 + \mathcal{T}_2 + \mathcal{R}_2 - \mathcal{T}_3 - \mathcal{R}_3 \\
N_3\dot{\mathbf{s}}_3 + \left\{\mathbf{s}_3,\, N_3\mathbf{s}_3\right\} &= \mathcal{G}_3 + \mathcal{T}_3 + \mathcal{R}_3 - \mathcal{T}_4 - \mathcal{R}_4 \\
N_4\dot{\mathbf{s}}_4 + \left\{\mathbf{s}_4,\, N_4\mathbf{s}_4\right\} &= \mathcal{G}_4 + \mathcal{T}_4 + \mathcal{R}_4 - \mathcal{T}_5 - \mathcal{R}_5 \\
N_5\dot{\mathbf{s}}_5 + \left\{\mathbf{s}_5,\, N_5\mathbf{s}_5\right\} &= \mathcal{G}_5 + \mathcal{T}_5 + \mathcal{R}_5 - \mathcal{T}_6 - \mathcal{R}_6 \\
N_6\dot{\mathbf{s}}_6 + \left\{\mathbf{s}_6,\, N_6\mathbf{s}_6\right\} &= \mathcal{G}_6 + \mathcal{T}_6 + \mathcal{R}_6
\end{aligned}
$$

So the cumulative equations get an extra term,

$$
\sum_{i=j}^{6}\left(\mathbf{z}_j^T N_i\dot{\mathbf{s}}_i + \mathbf{s}_i^T N_i[\mathbf{z}_j,\, \mathbf{s}_i] - \mathbf{z}_j^T\mathcal{G}_i\right) = \tau_j, \qquad j = 1, 2, \ldots 6
$$

Including Gravity continued

Can further incorporate gravity into the equations of motion using the "gravity twist",

$$\gamma = \begin{pmatrix} \vec{0} \\ -g\vec{k} \end{pmatrix}$$

so that

$$\mathcal{G}_i = N_i\gamma = \begin{pmatrix} -m_i g\vec{c}_i \times \vec{k} \\ -m_i g\vec{k} \end{pmatrix}$$

since gravity wrench is a pure force acting along a line through the centre of mass aligned with the negative z-direction.
Finally, equations of motion including gravity are then,

$$\sum_{i=j}^{6} \left(\mathbf{z}_j^T N_i(\dot{\mathbf{s}}_i - \gamma) + \mathbf{s}_i^T N_i[\mathbf{z}_j, \mathbf{s}_i] \right) = \tau_j, \qquad j = 1, 2, \ldots 6$$

Joint Angles

Can write equations in terms of the joint angels $\theta_1, \theta_2, \ldots, \theta_6$. Velocity twist of the ith link given by

$$\mathbf{s}_i = \dot{\theta}_1 \mathbf{z}_1 + \dot{\theta}_2 \mathbf{z}_2 + \cdots + \dot{\theta}_i \mathbf{z}_i = \sum_{j=1}^{i} \dot{\theta}_j \mathbf{z}_j$$

with $\mathbf{z}_i$ the ith joint twist as before. (Compare robot Jacobian).

Differentiating gives

$$\dot{\mathbf{s}}_i = \sum_{j=1}^{i} \ddot{\theta}_j \mathbf{z}_j + \sum_{1 \leq k < l \leq i} \dot{\theta}_k \dot{\theta}_l [\mathbf{z}_k, \mathbf{z}_l]$$

Generalised Mass Matrix

Can find explicit expressions for terms in usual presentation of the dynamics

$$\tau_i = A_{ij}\ddot{\theta}_j + B_{ijk}\dot{\theta}_j\dot{\theta}_k + C_i$$

For example, the generalised mass matrix A_{ij} is

$$A_{ij} = \begin{cases} \mathbf{z}_i^T(N_i + \cdots + N_6)\mathbf{z}_j, & \text{if} \quad i \geq j \\ \mathbf{z}_j^T(N_j + \cdots + N_6)\mathbf{z}_i, & \text{if} \quad i < j \end{cases}$$

Two Joint Robot

Can we diagonalise the mass matrix of a 2-joint-robot?
Mass matrix is

$$A = \begin{pmatrix} \mathbf{z}_1^T (N_1 + N_2)\mathbf{z}_1 & \mathbf{z}_1^T N_2 \mathbf{z}_2 \\ \mathbf{z}_2^T N_2 \mathbf{z}_1 & \mathbf{z}_2^T N_2 \mathbf{z}_2 \end{pmatrix}$$

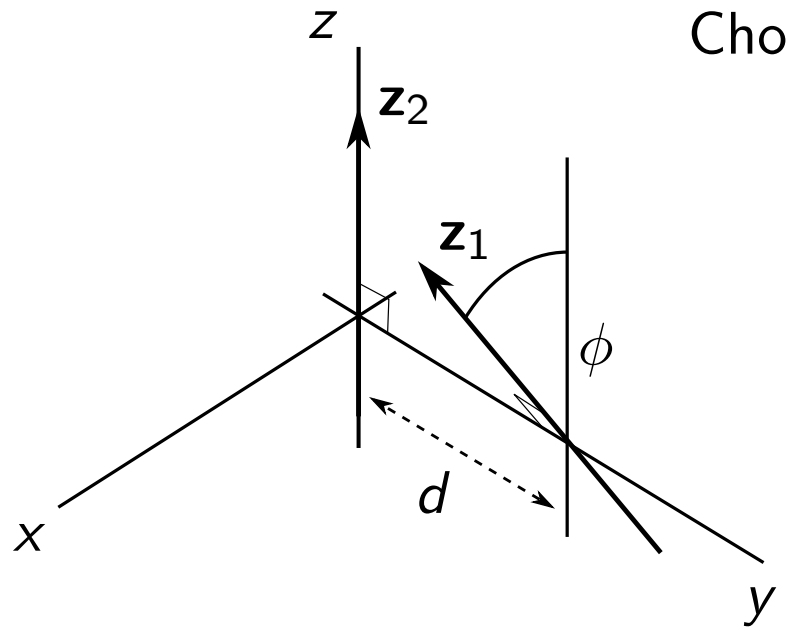
Can we choose joint axes so that $\mathbf{z}_1^T N_2 \mathbf{z}_2 = 0$ for all positions of the robot? Rodrigues formula for rotation about $\mathbf{z}_2$

$$\mathbf{z}_1^T \left(I_6 - \sin\theta_2 \,\mathrm{ad}(\mathbf{z}_2) + (1 - \cos\theta_2)\,\mathrm{ad}(\mathbf{z}_2)^2 \right)^T N_2 \mathbf{z}_2 = 0$$

For independence from θ_2 must have

$$\mathbf{z}_2^T N_2 \mathbf{z}_1 = 0, \quad \mathbf{z}_2^T N_2 [\mathbf{z}_2, \mathbf{z}_1] = 0 \quad \text{and} \quad \mathbf{z}_2^T N_2 \big[\mathbf{z}_2, [\mathbf{z}_2, \mathbf{z}_1]\big] = 0$$

Diagonalisation



Choose coordinates so that,

$$\mathbf{z}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } \mathbf{z}_1 = \begin{pmatrix} \sin\phi \\ 0 \\ \cos\phi \\ d\cos\phi \\ 0 \\ -d\sin\phi \end{pmatrix}$$

So that

$$[\mathbf{z}_2, \mathbf{z}_1] = \begin{pmatrix} 0 \\ \sin\phi \\ 0 \\ 0 \\ d\cos\phi \\ 0 \end{pmatrix}, \left[\mathbf{z}_2, [\mathbf{z}_2, \mathbf{z}_1]\right] = \begin{pmatrix} -\sin\phi \\ 0 \\ 0 \\ -d\cos\phi \\ 0 \\ 0 \end{pmatrix}, N_2\mathbf{z}_2 = \begin{pmatrix} \mathrm{I}_{13} \\ \mathrm{I}_{23} \\ \mathrm{I}_{33} \\ mc_y \\ -mc_x \\ 0 \end{pmatrix}$$

Diagonalisation continued

By inspection, wrenches dual to $\mathbf{z}_1$, $[\mathbf{z}_2, \mathbf{z}_1]$ and $\big[\mathbf{z}_2, [\mathbf{z}_2, \mathbf{z}_1]\big]$ are

$$\mathcal{W}_\lambda = \begin{pmatrix} -d\cos\phi \\ 0 \\ 0 \\ \sin\phi \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{W}_\mu = \begin{pmatrix} 0 \\ -d\cos\phi \\ 0 \\ 0 \\ \sin\phi \\ 0 \end{pmatrix}, \quad \mathcal{W}_\nu = \begin{pmatrix} 0 \\ 0 \\ d\sin\phi \\ 0 \\ 0 \\ \cos\phi \end{pmatrix}$$

So we must have

$$N_2\mathbf{z}_2 = \begin{pmatrix} \mathrm{I}_{13} \\ \mathrm{I}_{23} \\ \mathrm{I}_{33} \\ mc_y \\ -mc_x \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} -d\cos\phi \\ 0 \\ 0 \\ \sin\phi \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -d\cos\phi \\ 0 \\ 0 \\ \sin\phi \\ 0 \end{pmatrix} + \nu \begin{pmatrix} 0 \\ 0 \\ d\sin\phi \\ 0 \\ 0 \\ \cos\phi \end{pmatrix}$$

For some constants λ, μ and ν.

Diagonalisation continued

From last row, have $\nu \cos\phi = 0$. Either $\nu = 0$ or $\phi = \pm\pi/2$. Can't have $\nu = 0$, that would make $\mathrm{I}_{33} = 0$ - integral of density and squares of distances (third row). So $\phi = \pm\pi/2$. Implies perpendicular joint axes.
Setting $\phi = \pi/2$ gives

$$N_2\mathbf{z}_2 = \begin{pmatrix} 0 \\ 0 \\ \nu d \\ \lambda \\ \mu \\ 0 \end{pmatrix} \quad \text{so that} \quad \mathbf{z}_2^T N_2 Q_0^* N_2 \mathbf{z}_2 = 0$$

The wrench is a pure force. Implies $\mathbf{z}_2$ is a percussion axis for N_2.
If, when a rigid body is subject to an impulsive pure force it begins to move as an instantaneous pure rotation, the axis of this rotation is said to be a percussion axis for the body. These axes form a tetrahedral line complex — the complex of normals to the isogyre quadrics.

Selig, J. M. and Martin D.,2014,"On the line geometry of rigid-body inertia." *Acta Mechanica*, **225**(11):3073—3101.

Conclusions

- Extends to tree structured mechanisms and parallel robots.
- Can derive same equations using Lagrangian dynamics.
- Only rigid-body dynamics, can include friction and compliance.

THANK YOU