Number Systems, and Just Touching Covering Systems

- Expressing numbers in various bases, with unusual coefficients…

Writing numbers in bases

- N=3 and A={0,1,2}

- 34 = 1021 base 3
- $34 = 1(3^0)+2(3^1)+0(3^2)+1(3^3) = \Sigma\, a_i N^i$

- We say 34 is expressible in A
- 34 is in E_A

Finding the expression

- A= {0,1,2}   34 = 1021 base 3
- $34 = 1+2(3)+0(3^2)+1(3^3)$
- $F_A : x \longrightarrow (x-a)/3$ for suitable a
- $34 \xrightarrow{1} 11 \xrightarrow{2} 3 \xrightarrow{0} 1 \xrightarrow{1} 0$

The way the map works

- $34 = 1+2(3^1)+0(3^2)+1(3^3)$
- $11 = 2+0(3^1)+1(3^2)$
- $3 = 0+1(3^1)$
- $1 = 1$
- $0 = 0$

Not every number reaches 0

- $-34 \xrightarrow{2} -12 \xrightarrow{0} -4 \xrightarrow{2} -2 \xrightarrow{1} -1 \xrightarrow{2} -1$

- -34 has no expression, is not in E_A
- (obvious anyway since A has no negative numbers)

Graph of A

- Connect all integers x to $F_A(x)=(x-a)/N$
- If *all* numbers connect to 0, E_A = Z and A is called a *number system*
- All integers connect eventually to a ‘small integer’ in **I**, the ‘interval for A’
- Actually need to draw the graph only on **I**

Graph of A example 1

Figure 1: A = {0,-5,11} where I = [-5,5]

4 → 3 → 1 → 2 → -3 → -1 → -4 → -5 → 0 ↺

5 → -2 ↗ 1

Periodic Numbers {0}

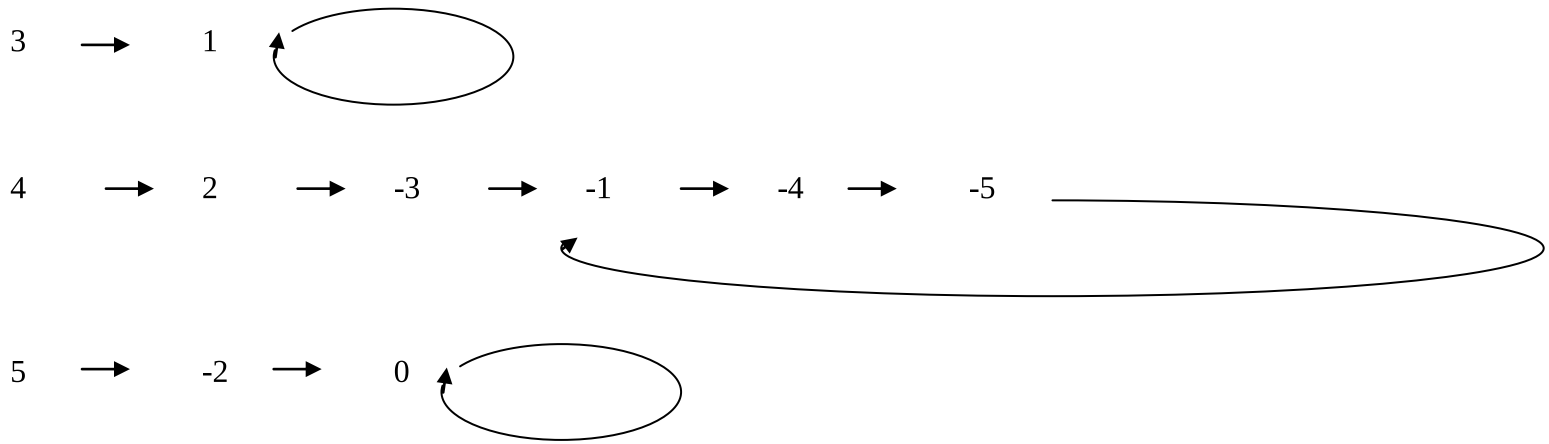
Express 2 in {0,-5,11}

- A = {0,-5,11}

- $2 = 11+0(3)+11(3^2)+11(3^3)+-5(3^4)$

Graph of A example 2

Figure 2: A = {0,-2,11} where I = [-5,5]



Periodic Numbers {1, -1, -4, -5, 0}

Example 3: A = {0,13,*}

$$-2 \xrightarrow{13} -5 \xrightarrow{13} -6 \xrightarrow{0} -2 \xrightarrow{13} -5 \text{ etc}$$

A = {0,13,*}:

E_A is not Z, regardless of the other number:

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Problems with A

- A needs a mix of signs
- A cannot have 13 if N=3
- A must have a small number (in **I**): if A = {0,-10,13} , the set **I** is [-6,6] and none of the numbers in **I** apart from 0 are expressible

Switch to B

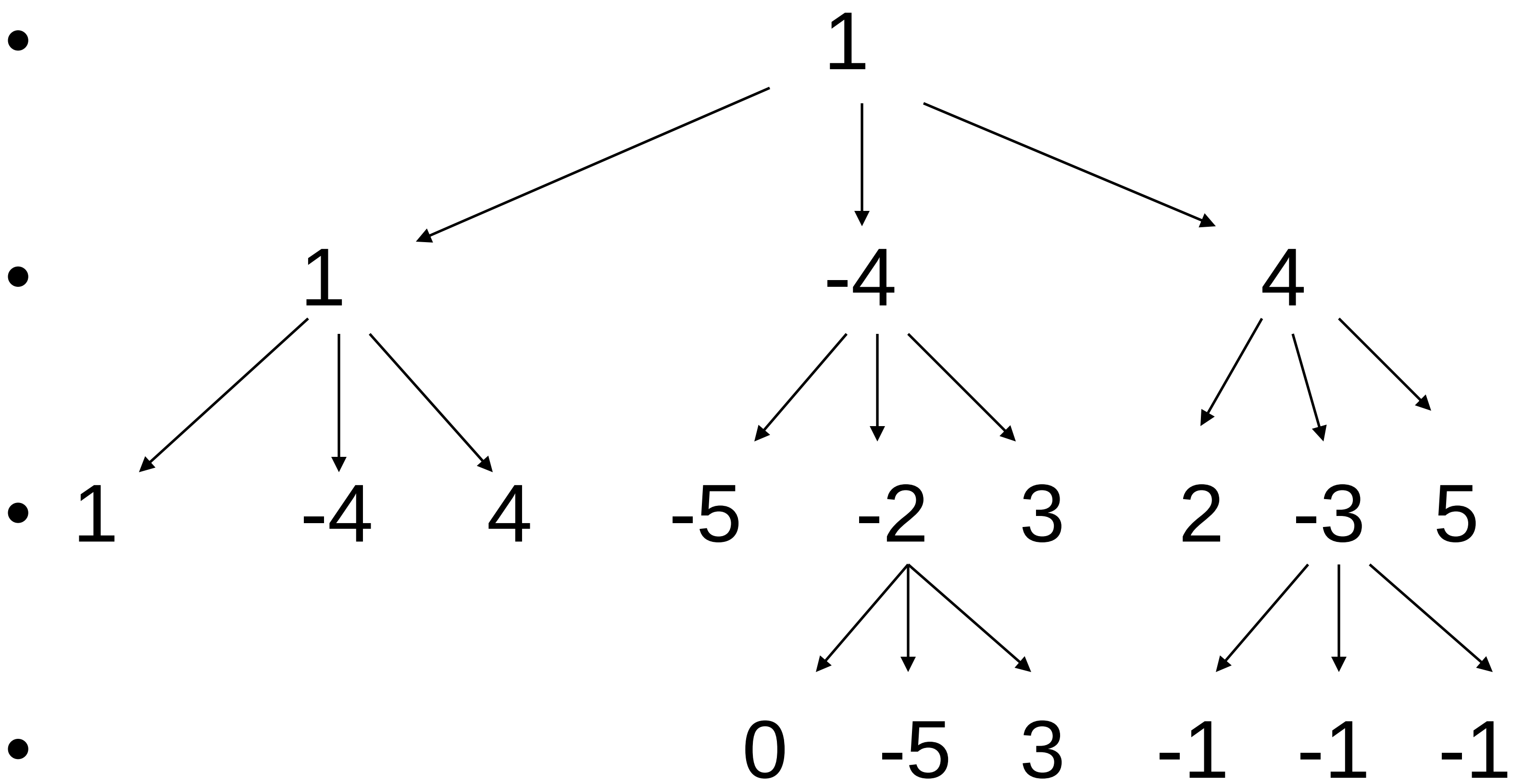
- For A={0,-10,13}, let B=A-A
- Note B will always contain 0; other As can create the same B
- B will always have a mix of signs
- In the example B={0,-10,10,13,-13,23,-23} but the **I** for B is bigger : **I** = [-11,11]
- If A = {0,10,11}, B= {0,10,-10,11,-11,1,-1} has the same **I** but has a small number.

B=A-A

- N=3, A={0,1,2}, B={-2,-1,0,1,2}
- E_B= Z

- N=3, A={0,-2,11}, B={0,2,-2,11,-11,13,-13}
- E_B= Z
- If E_B= Z , A is called a Just Touching Covering System

A={0,-2,11} and B={0,2,-2,11,-11,13,-13}



- $1 = 13+2(3)+-2(3^2)$

Maps used in tree

- $A=\{0,-2,11\}$
- Maps used (in order) are:
- $F1(x)=(x+0-a)/3$ (eldest child)
- $F2(x)=(x+-2-a)/3$ (middle child)
- $F3(x)=(x+11-a)/3$ (youngest child)

- Note: sum of children = parent

E_B=Z?

- To decide if E_B=Z, draw the graph of B on the interval **I**, and see whether all numbers connect to 0
- Messy, but doable

THEOREM

- If for ex. A={0,5,10}, then every member of B is divisible by 5, as is every member of E_B
- Therefore, if E_B=Z, you need gcd(B) =1
- In fact gcd(B)=1 is all you need for E_B=Z:

Theorem 1 (2001)

- N real integer, abs(N)>1
- $A=\{a_0,a_1,a_2,\ldots,a_{N-1}\}$ with $a_i \equiv i \pmod N$
- B= A-A
- Theorem: $E_B = \langle d \rangle$ where d = gcd(B)
- Corollary: $E_B=Z$ *iff* gcd(B)=1
- Much easier to decide using the gcd than the graph

Original Proof:

- *Base N Just Touching Covering Systems* (Publicationes Mathematicae Debrecen, Tomus 58/4 (2001), pp. 549-557)

Apply Theorem

- Example 1
- N = 3, A={0,2,7}
- gcd(B) = 1 , so E_B=Z

- Example 2
- N=6, A={0,7,-10, 15, 10, -631}
- gcd(B) = 1, so E_B=Z

Idea of Proof

- Additive Structure of Trees
- N=3, A ={0,2,7}, B={0,2,-2,5,-5,7,-7}

1	**2**	**3**
-2 1 **2**	0 **-1** 3	1 1 **1**
-3 0 1 -2 1 2 **0** -1 3	0 0 0 1 **-2** 2 1 1 1	-2 1 2 -2 1 2 **-2** 1 2

- E_B turns out to be a group (the group <d>), but it is hard to show it is a group directly (hard to show that the sum of expressible numbers is expressible)

How maps add:

- x maps to (x+a1-a2)/N
- y maps to (y+a2-a3)/N
- x+y maps to (x+y+a1-a3)/N
- Note x and x+y use the same map

Group Structure

- To show E_B is the group <d>, show the sum of expressible numbers is expressible.
- You can prove this if you know *one* nonzero number x is *omniexpressible (*this means each descendent of x is also expressible; it also means there are an infinite number of expressions for x: eg any $b_1+b_2N+b_3N^2$ congruent to x mod N^3 can be extended to an expression for x)

The set of omniexpressibles

- Let H be the set of omniexpressible numbers, which is a *group*:
- If x is omni, then so is –x
- If x and y are omni then so is x+y
- The crux of the proof is showing H is not just the group {0}

Proof

- This is the proof assuming H is not just {0}

Coset Illustration A={0,4,8}

- 0+H = {…,-12, -8, -4, 0, 4, 8, 12,…} = H
- 1+H = {…,-11, -7, -3, 1, 5, 9, 13,…}
- 2+H = {…,-10, -6, -2, 2, 6,10,14,…}
- 3+H = {…,-9,   -5, -1, 3, 7,11,15,…}
- The cosets partition Z
- For each coset, the numbers contained are either all expressible or all inexpressible:

Coset Type

- Consider the set of cosets of H:
- If x is expressible so is any member of x+H
- If x is inexpressible so is any member of x+H
- Thus there are two types of cosets: *expressible* and *inexpressible* (0+H is of course an expressible one)

explanation

- If x is expressible, so is x+h:
- x+h maps to 0+h' under some F, and from there to 0+0 (since h' is expressible as are all the descendents of h)
- If x is inexpressible, so is x+h:
- If x+h maps to 0 under some F, then x must map to t and y must map to –t under two paths that add to F. But this is impossible as t cannot be both inexpressible and expressible.

Maps and Cosets

- The basic maps used in creating the tree of descendents of x can be denoted as: $F_{A\text{-}ai}$ where a_i is in A: $F_{A\text{-}ai}(x) = (x+a_i-a)/N$ for the appropriate a in A
- A general path through the tree of x can be described as connecting x to F(x) where F is a composition of such maps
- It is easy to show F(x+H)=F(x)+H: from an earlier slide F(x+h)=F(x)+F'(h)=F(x)+h'

Example (middle child)

- A={0,4,20} with $F_{A-4}(x) = (x+4-a)/3$
- H= { …,-8, -4, 0, 4, 8, 12, 16,…}
- 1+H= {…, -7, -3, 1, 5, 9, 13, 17,…}

- F(1+H) = {…, -1, -1, -5, 3, 3, -1,7,…}
- F(1)+H = -5+H = same set

Images of cosets are cosets

- Z is the union of the cosets 0+H, x_1+H,…,x_k+H (choosing a set of names for the cosets).
- For any map F as described, F(Z)=Z
- That is, F is an onto map
- F(Z) = union of F(x_i +H)=Z
- Since F(x+H) = F(x)+H , Z is the union of the cosets F(0)+H, F(x_1)+H,…,F(x_k)+H ,the same cosets in possibly a different order.

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Each F respects expressibility

- If x is inexpressible so is F(x) , so inexpressible cosets map to inexpressible cosets since F(x+H) = F(x)+H
- Because there are a FINITE number of cosets, expressible cosets must map to expressible cosets!
- If x+H is an expressible coset, so must be F(x+H) = F(x)+H. If x is expressible, so is F(x) ,so *all expressible x are in fact omni*! Thus E_B = H, a group.

E_B is <d> where d = gcd(B)

- If gcd(B)=d then every member of E_B is a multiple of d. In fact E_B includes *all* the multiples of d:
- The members of B itself are expressible (in one step) and so is any sum of these b's, including d itself.
- (example B={0,4,-4,14,-14,10,-10}:d=2 and 2 = 4+4+4+-10)
- Since d is in E_B, all multiples of d are as well.

The Last Part

- We need to show H is not {0} ie that there is a nonzero omniexpressible number:
- If H is only {0} ,the proof above does not work: The cosets of H consist of the sets of size one, eg 3+H ={3}. Say the inexpressible cosets were the odd numbered ones and the expressible ones were the evens; It could happen that F maps inexpressibles to inexpressibles but expressibles to cosets of either type:

Why you need a finite number of cosets:

- F could take coset {x} to {x} if x is odd (so inexpressible cosets go to inexpressible ones) but could do this to the expressible ones:
- F({2})={1}, F({4})={2}, F({6})={3} etc
- So some expressible cosets map to inexpressible cosets, and F is still *onto* from Z to Z

- Now show H, the set of omniexpressible numbers, is not just the set {0}

Subtraction of Trees

- If you take any two numbers x and y and a single F as described above, F(x)-F(y) is equal to F'(x-y) for some F'
- Therefore when you subtract numbers, you subtract the paths as well (the *same* path for each):

Subtraction via *the same* path

- N=3, A ={0,25,5}, B={0,5,-5,20,-20,25,-25}
- **5** **4** **1**
- 0 **10** -5 -7 **8** 3 -8 7 **2**
- 0 0 0 **-5** 10 5 -10 5 0 -4 6 -9 **1** 11 -4 1 1 1 -11 4 -1 -6 9 4 -1 9 **-6**

Why:

- Define $F(z) = (z+a1-a)/N$ for appropriate a
- $F(x)=(x+a1-a2)/N$
- $F(y)=(y+a1-a3)/N$
- $F(x)-F(y) = (x-y+a3-a2)/N$
- This equals $F'(x-y)$ where

$F'(z)=(z+a3-a)/N$ for appropriate a

“merging”

- N=3, A ={0,25,5}, B={0,5,-5,20,-20,25,-25}
- **14** **4** **10**
- 3 13 -2 -7 8 3 -5 10 5
- 1 1 1 -4 11 6 -9 6 1 -4 6 -9 1 11 -4 1 1 1 -10 5 0 -5 10 5 0 10 -5

- Conclusion: x-y connects to 0 if and only if $F(x)=F(y)$ for some F
- If such an F exists we say x and y merge.
- If x and y merge, x-y is expressible and conversely.

Maximal nonmerging sets

- Let M be a set of numbers which do not merge pairwise
- $M=[x_1,x_2,x_3,\ldots.,x_k]$ (the x_i increase)
- Also let M be an example of the largest possible such set (If M has more terms than **I** does, it must have two that merge)
- By maximality of M, any number not in M must merge with a member of M

Maps on M

- If $M=[x_1,x_2,x_3,\ldots,x_k]$ is a MNMS, so is $F(M)=[F(x_1), F(x_2),\ldots,F(x_k)]$:
- Obviously F(M) is the same size since there is no merging
- If $F(x_1)$ and $F(x_2)$ merged that would mean x_1 and x_2 merge

Goal

- If there are two examples of M which differ in only one term, then we have the omni number:
- $M1=[x_1,x_2,x_3,\ldots.,x_k]$
- $M2= [x_1,x_2,x_3,\ldots.,x_k']$
- Then $x_k - x_k'$ is omni and nonzero:

Why?

- $M1=[x_1,x_2,x_3,\ldots,x_k]$, $M2=[x_1,x_2,x_3,\ldots,x_k']$
- x_k-x_k' goes to 0 by maximality
- If you apply *any* F to M1 and M2 you get

$F(M1) = [F(x_1), F(x_2),\ldots,F(x_k)]$ and

$F(M2) = [F(x_1), F(x_2),\ldots,F(x_k')]$

so $F(x_k)-F(x_k')$ also goes to 0 by maximality;

And $F(x_k)-F(x_k')$ is an *arbitrary* descendent of x_k-x_k'

Last Step: Find two such Ms

- Take any M and let it lie in a interval [a,b] containing **I**
- There are an infinite number of Fs so two of them (F1 and F2) must have the same action on [a,b] since each F is a map from [a,b] to [a,b].
- It is not hard to show that F1 and F2 do not agree on the whole of Z :

Example:how you show maps are all different

- F_{A-ai} where a_i is in A: $F_{A-ai}(x) = (x+a_i-a_j)/N$ for the appropriate a_j in A
- Say $F_{A-a1}(x) = F_{A-a2}(x)$ for all x
- WLOG let A consist of 0 and other positive numbers, and say a1 > a2.
- Let x = N(a2)-a1
- $F_{A-a1}(x)$=a2 and
- $F_{A-a2}(x)$ = (N(a2)-a1+a2-a3)/N=
- =a2+(a2-a1-a3)/N
- so a2=a1+a3, a contradiction

Example of how I finish the proof:

- A={0,5,25}
- M={-2,-1,0,1,2} is an MNMS
- F1(x)=(x+0-a)/3 = F_A
- F2=$F1^5$ (F1 applied 5 times)
- F1(M)=F2(M)={-4,-2,0,-3,-1} (see next slide)

- A={0,5,10}   M={-2,-1,0,1,2} F_A does this:
- -5 $\longrightarrow$ -5 $\longrightarrow$ -5…
- 5 $\longrightarrow$ 0 $\longrightarrow$ 0…
- 4 $\rightarrow$ -2 $\rightarrow$ -4 $\rightarrow$ -3 $\rightarrow$ -1 $\rightarrow$ -2 $\rightarrow$ -4…
- 3 $\rightarrow$ 1 $\rightarrow$ -3 $\rightarrow$ -1 $\rightarrow$ -2 $\rightarrow$ -4 $\rightarrow$ -3…
- 2 $\rightarrow$ -1 $\rightarrow$ -2 $\rightarrow$ -4 $\rightarrow$ -3 $\rightarrow$ -1 $\rightarrow$ -2…
- So $F_A(M) = F_A^5(M)$

- F1(3)= 1 but F2(3)=-4
- M redefine as M+1={-2,-1,0,1,2}+1
- M={-1,0,1,2,3}
- F1(M)= {-2,0,-3,-1,1}
- F2(M)={-2,0,-3,-1,-4}
- Then -4-1=-5 is omni

Conclusion:

- Let s be the first number bigger than b on which F1 and F2 do not agree
- $M' = [x_1, x_2, \ldots., x_k] + s - x_k = [y_1, y_2, \ldots., s]$ is still a maximal nonmerging set.
- Recall M is in increasing order so the y_i are smaller than s
- All numbers except s satisfy $F1(y_i)=F2(y_i)$
- Then F1(M') and F2(M') differ only in their last term (F1(s) vs F2(s)) QED

Unobvious corollary of proof

- All expressible numbers are in fact omni
- If you have an A and form A-A=B, the set B can be created by other sets A+k
- The same E_B can be created from $B^r=A^r-A^r$ as well (using N^r instead of N); A^r consists of sums from the N^0 to N^{r-1} term with coefficients in A.
- It is a consequence of the omniexpressible part of the proof that there is an A^r for which $E_Br = E_Ar$: in other words there is a *specific* path to 0 for every expressible number: see the following example.

Example A={0,2,7}

- Because of omniness, you can find a single path via which *all* in **I** go to 0
- $F1(x)=(x+2-a)/3$ $F2(x)=(x+7-a)/3$
- $F3(x)=(x+0-a)/3$ $F4(x)=(x+2-a)/3$
- $F(x) = F4 \circ F3 \circ F2 \circ F1$
- -3 -2 -1 0 1 2 3
- -1 0 -2 0 1 -1 1
- 2 0 1 0 2 2 2
- 0 0 -2 0 0 0 0
- 0 0 0 0 0 0 0

An A that replaces B

- Then by applying F repeatedly to any expressible x you will eventually reach 0:
- The coefficients used come from subtracting members of the 4 shifted sets in order: A1= A-2 ={-2,0,5},

  A2=A-7={-7,-5,0}, A3=A-0={0,2,7}, A4=A-2={-2,0,5}

$100 = -2+-5(3)+7(3^{2})+ 5(3^{3})+ 5(3^{4})+ -5(3^{5})+ 7(3^{6})+ -2(3^{7})$

$100= (-2+-5(3)+7(3^{2})+ 5(3^{3}))+(5-5(3^{1})+ 7(3^{2})+ -2(3^{3}))(3^{4})$

Equivalent to using N = 81 and A2 where

A2= {-2,0,5}+3{-7,-5,0}+9{0,2,7}+27{-2,0,5} , a set of 81 numbers, one of each congruence type

The end

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