Exploring the Zeta function and Riemann Hypothesis

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Abstract

The Riemann Hypothesis is one of the most profound and challenging unsolved problems in mathematics, originating from Bernhard Riemann's 1859 conjecture on prime number distribution. Central to number theory, this hypothesis has implications for prime distribution, quantum physics, and cryptography. It is closely tied to the zeros of the zeta function in the complex plane.

Keywords: Prime numbers distribution, Zeta function, Riemann Hypothesis.

1 Introduction

In 2000, the Clay Mathematics Institute established the Millennium Prize Problems, listing seven most challenging and profound unsolved mathematical problems. Among these is the Riemann Hypothesis, stemming from German mathematician Bernhard Riemann's 1859 conjecture on the distribution of prime numbers, which remains unsolved. This hypothesis remains central to number theory, with implications for prime number distribution and cryptography. The Riemann Hypothesis is comparable in the most renowned mathematical challenges, drawing global interest following the resolution of Fermat's Last Theorem by Andrew Wiles and Richard Taylor in the mid-1990s, though Fermat's theorem itself is not a Millennium Problem.

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2 The Prime Number Theorem

Let $\mathbb{P}$ be the set of prime numbers, and for any $x \in \mathbb{R}$, we define $\pi(x)$ the prime-counting function that gives the number of primes less than or equal to a real number x, by:

$$\pi(x) = \sum_{p \in \mathbb{P} | p \leq x} 1 \tag{1}$$

the number of prime numbers less than or equal to x.

After series of numerical experiments, Carl Gauss conjectured in 1792 or 1793, at age 15 or 16, that when x tends to $+\infty$ the function $\pi(x)$ is asymptotically equivalent to $\frac{x}{ln(x)}$. Later in 1896 that conjecture has been independently proven by J. Hadamard and C.J de la Vallée-Poussin and since it has been known as the **Prime Number Theorem**.

Theorem 1 (The prime number theorem)**.** *We have:* $\pi(x) \sim \frac{x}{ln(x)}$ *when* $x \to +\infty$.

3 Riemann Zeta function

Mathematicians have studied the Riemann zeta function for centuries due to its profound outputs and its intricate encoding of prime number distribution. This function not only yields complex mathematical insights but also has significant applications in physics, including quantum mechanics and statistical mechanics, highlighting its interdisciplinary importance.

The Riemann zeta function is defined as the sum of the reciprocal of the natural numbers raised to an exponent s.

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + ... \tag{2}$$

Through analytical continuation, mathematicians have extended the domain of the zeta function to all complex values of s, allowing for an in-depth exploration of its properties. This method effectively broadens the function's original scope. In 1650, the mathematician **Pietro Mengoli** posed the foundational question of finding the exact value of $\zeta(s)$ at $s = 2$, which later became crucial for understanding the series convergence in the study of the zeta function.

If $s = 2$ the series

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + ... \tag{3}$$

converges, in other words, the exact sum of the reciprocal of the squares of all natural numbers, known as **Basel's problem**, has a finite value.

Let us consider the sequence of partial sums, where S_n represents the sum of the first n terms of the series $\zeta(2)$

$$S_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ... + \frac{1}{n^2} \tag{4}$$

Or, equivalently,

$$S_n = \sum_{k=1}^{n} \frac{1}{k^2} \tag{5}$$

Consequently, we have

$$\lim_{n \to +\infty} S_n = \lim_{n \to +\infty} \sum_{k=1}^{n} \frac{1}{k^2} = \zeta(2) \tag{6}$$

For each integer n, it corresponds a value S_n, as n increases, S_n tends to a certain number. That number, to which S_n converges when n tends to infinity, is called the limit $\zeta(2)$, which is finite.

However, if $s = 1$ the series diverges to $+\infty$, indicating that the sum does not converge to a finite value when $s = 1$. Thus the series

$$\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ... \tag{7}$$

diverges to infinity.

Proposition 1. *The sequence $\zeta(1)$ diverges towards infinity.*

$\lim_{n \to +\infty} \sum_{k=1}^{n} \frac{1}{k} = +\infty$

Proof. (By contradiction). Let us assume that limit when tends to infinity has a finite value S.

$\sum_{k=1}^{n} \frac{1}{k} = S \quad \Leftrightarrow \quad S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ...$

If we multiply S by $\frac{1}{2}$, we obtain:

$\frac{1}{2}S = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + ...$

If we subtract $\frac{1}{2}S$ from S, we get:

$\frac{1}{2}S = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + ...$

The difference of two expressions is

$0 = \frac{1}{2} + \frac{1}{12} + \frac{1}{30} + \frac{1}{56} + ... > 0.$

Contradiction. The limit S cannot be finite. □ □

In 1736, Leonhard Euler solved the **Basel problem**, earning him immediate fame and recognition within the mathematics community. This achievement was not Euler's only significant contribution through his study of the zeta function. In addition to solving the Basel problem, he also managed to express the zeta function as an infinite product, revealing deep connections to prime numbers.

$$\zeta(s) = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1} \tag{8}$$

The Riemann zeta function can be expressed as a product, known as the Euler product formula, discovered in 1737. It shows the first relationship between the zeta function and prime numbers. Modern study of prime numbers has always been based on the zeta function.

$$\sum_{n=1}^{+\infty} \frac{1}{n^s} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1} \tag{9}$$

Euler's product formula presents an alternative expression of the zeta function while revealing its profound link with prime numbers. Notably, this formula aids in calculating the probability that two randomly chosen integers are relatively prime, thereby bridging essential concepts in number theory.

Let a and b be two randomly chosen positive integers. What is the probability that they are relatively prime (having no common factors other than 1)? Otherwise $Prob(gcd(a,b) = 1)$.

What is the probability that a and b are not divisible by 2? That probability would be $\left(1 - \frac{1}{2^2}\right)$.
What is the probability that a and b are not divisible by 3? That probability would be $\left(1 - \frac{1}{3^2}\right)$.
What is the probability that a and b are not divisible by 2 or 3? That probability would be $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)$.
One can follow the same pattern with being not divisible by 3, 5, 7 and so on. Extending the process indefinitely yields an infinite product of $\left(1 - \frac{1}{p^2}\right)$, where p represents all prime numbers from 2 to infinity, thus capturing the fundamental role of primes in the function structure.

$$Prob(gcd(a,b) = 1) = \prod_{p\ prime} \left(1 - \frac{1}{p^2}\right) = \prod_{p\ prime} \left(\frac{1}{1 - \frac{1}{p^2}}\right)^{-1} = \frac{1}{\zeta(2)} \qquad (10)$$

In his famous paper in 1859 Figure 1, Riemann explained that the variable s is not restricted to natural numbers; it can extend to all real numbers. Furthermore, s can also take complex values, broadening the function domain significantly. Riemann demonstrated the close connection between the distribution of prime numbers and the zeros of $\zeta(s)$ in the complex plane.

Hence, the link between the analytical properties of the function ζ and the distribution of prime numbers is defined through its expression as a Eulerian product.

$$\zeta(s) = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1} \qquad if \qquad \Re(s) > 1 \qquad (11)$$

The Riemann zeta function is an important function in analytic number theory and the study of prime numbers. The Riemann hypothesis examines zeros outside the region of convergence of the series.

The proof of the Prime Number Theorem is based on the properties of the Riemann zeta function for complex values of the argument s. Recall that this function is defined on the half-plane $\Re(s) > 1$ by the following formula:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \tag{12}$$

A key point in the proof by Hadamard and Vallée-Poussin is that the zeta function $\zeta(s)$ admits a meromorphic extension in a neighbourhood of the closed half-plane $\Re(s) \geq 1$, with a single pole at $s = 1$ and no zeros along line $\Re(s) = 1$. Riemann had previously established in his 1859-paper the existence of a meromorphic extension of the function $\zeta(s)$ over the entire complex plane $\mathbb{C}$ where he also highlighted the close connection between prime number distribution and the zeros of $\zeta(s)$ in the complex domain.

By that time, the study of prime number distribution had already inspired significant work, both rigorous and conjectural, that did not rely on complex analysis methods. In 1737, Euler used the function ζ, as a function of a real variable, to investigate the sequence of prime numbers. The prime number theorem's equivalent for $\pi(x)$ had been conjectured in the late 18th Century by Gauss and Legendre. Moreover, shortly before Riemann's 1859-paper, Chebyshev had established, using elementary methods, the existence of two positive constants A and B with $0 < A < 1 < B$, such that for sufficiently large numbers x, we have the following double inequality.

$$A.\frac{x}{ln(x)} \leq \pi(x) \leq B.\frac{x}{ln(x)} \tag{13}$$

In fact, Chebyshev proved that

$$A = \frac{ln(2)}{2} + \frac{ln(3)}{3} + \frac{ln(5)}{5} - \frac{ln(30)}{30} \approx 0.92129$$

$$\text{And} \qquad B = \frac{6A}{5} \approx 1.10555$$

Over years these bounds have been improved, for example Sylvester in 1892 showed that:

$$0.956\frac{x}{ln(x)} \leq \pi(x) \leq 1.045\frac{x}{ln(x)} \tag{14}$$

However, a complete proof of the Prime Number Theorem was only achieved at the late 19th Century, after advancements made in complex function theory, which provided the necessary tools to rigorously establish the theorem.

Riemann demonstrated how to extend the zeta function to all possible values in $\mathbb{C}$, except 1. So, there is only one value where the function is not defined, that case is called a **pole** or a **singularity** corresponding to $s = 1$, where the zeta function cannot be extended.

Thus, Riemann's hypothesis concerns the zeros of the zeta function, specifically the values of the variable s for which $\zeta(s) = 0$. For all negative even integers, the zeta function $\zeta(s)$ equals 0; these values are known as the **trivial zeros** of the zeta function. The question then arises: does the function $\zeta(s)$ have other zeros?

Observations indicate that all additional **non-trivial** zeros appear to lie within a specific region called the **critical strip**. This strip is bounded by two vertical lines, namely the imaginary axis and the line $x = 1$, containing all complex numbers where the real part lies between 0 and 1. More precisely, the middle line represented by $x = \frac{1}{2}$, contains these zeros. Huge number of zeros have been found on this line, like trillions of numbers.

In fact, the Riemann Hypothesis asserts that all non-trivial zeros of the zeta function lie exactly on this line Figure 2.

4 Conclusion

The prime counting function $\pi(x)$ introduced by Gauss in the 18th Century estimates the number of primes less than a given value x. Gauss proposed that $\pi(x)$ could be approximated by $\frac{x}{ln(x)}$. This result is later formalised as the prime number theorem. In his 1859 paper, Riemann advanced this study by introducing the zeta function, thereby linking it intrinsically to prime distribution. He further extended the zeta function's domain from real numbers to the whole complex plane.

The Riemann zeta function has two types of zeros: trivial zeros at all negative even integers, and non-trivial zeros whose distribution remains central to number theory. The Riemann Hypothesis posits that all non-trivial zeros lie on the line with real part $x = \frac{1}{2}$. Until today, the Riemann hypothesis remains unproven and deeply influential in the field of mathematics and number theory.

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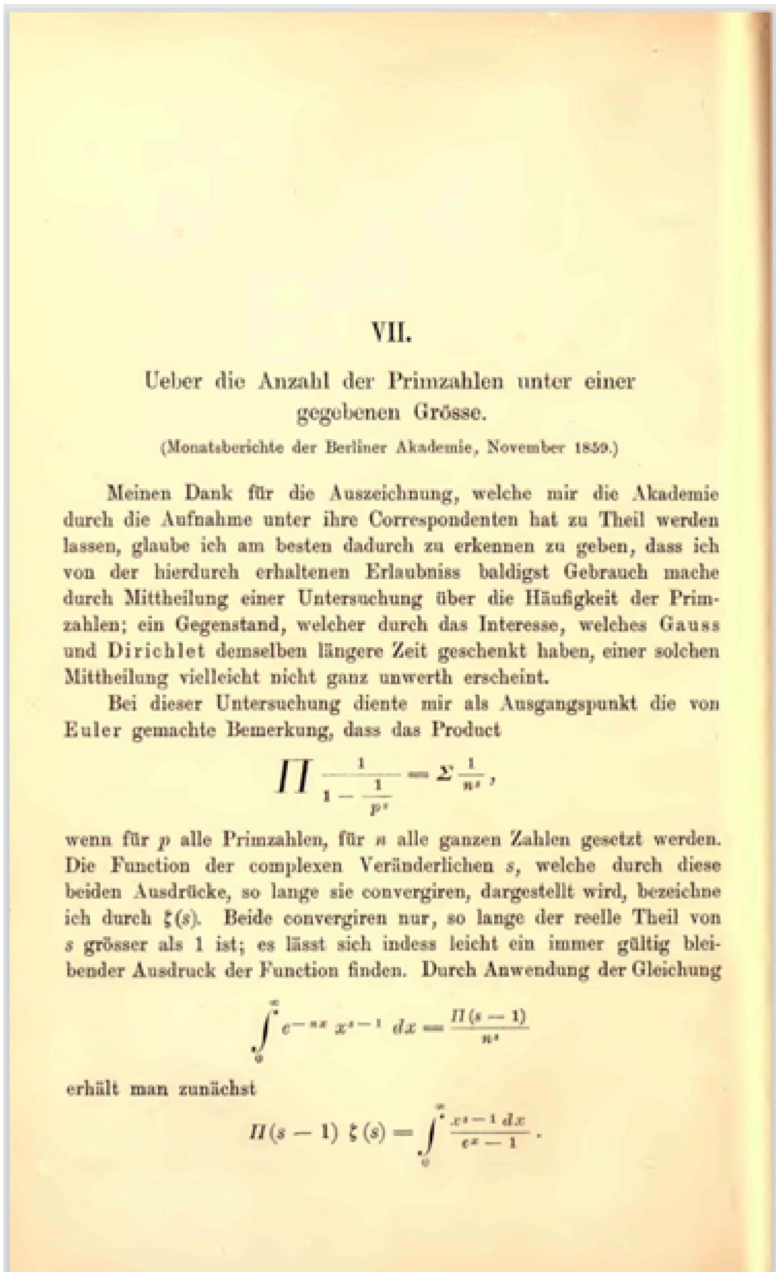

VII.

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Monatsberichte der Berliner Akademie, November 1859.)

Meinen Dank für die Auszeichnung, welche mir die Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaube ich am besten dadurch zu erkennen zu geben, dass ich von der hierdurch erhaltenen Erlaubniss baldigst Gebrauch mache durch Mittheilung einer Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch das Interesse, welches Gauss und Dirichlet demselben längere Zeit geschenkt haben, einer solchen Mittheilung vielleicht nicht ganz unwerth erscheint.

Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Product

$$\Pi \frac{1}{1-\frac{1}{p^s}} = \Sigma \frac{1}{n^s},$$

wenn für p alle Primzahlen, für n alle ganzen Zahlen gesetzt werden. Die Function der complexen Veränderlichen s, welche durch diese beiden Ausdrücke, so lange sie convergiren, dargestellt wird, bezeichne ich durch $\zeta(s)$. Beide convergiren nur, so lange der reelle Theil von s grösser als 1 ist; es lässt sich indess leicht ein immer gültig bleibender Ausdruck der Function finden. Durch Anwendung der Gleichung

$$\int_0^\infty e^{-nx} x^{s-1}\, dx = \frac{\Pi(s-1)}{n^s}$$

erhält man zunächst

$$\Pi(s-1)\, \zeta(s) = \int_0^\infty \frac{x^{s-1}\, dx}{e^x - 1}.$$

Figure 1: Riemann paper.

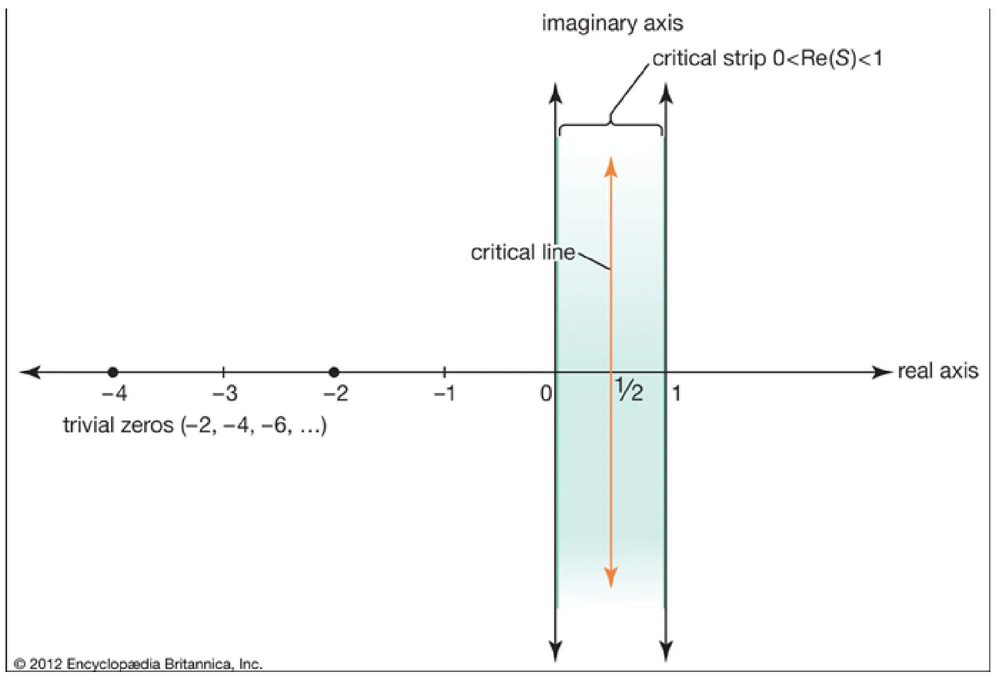

Figure 2: Critical strip.