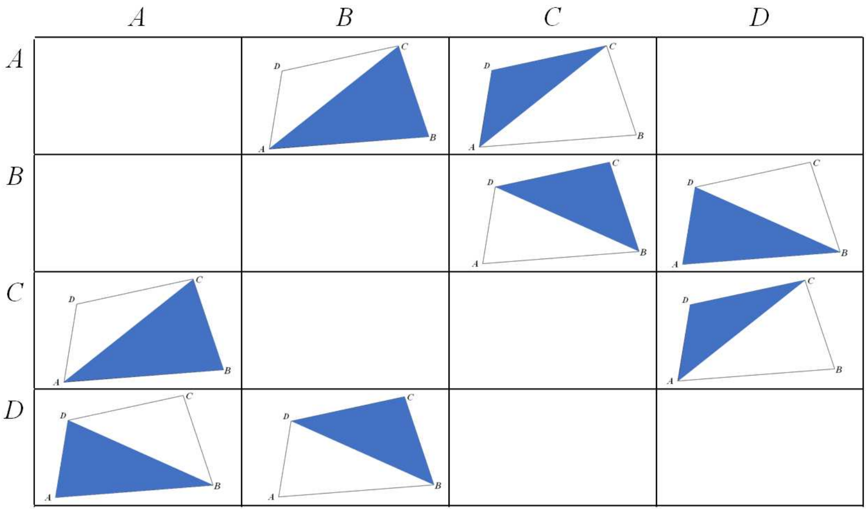
A Matrix of Triangle Areas Part 2



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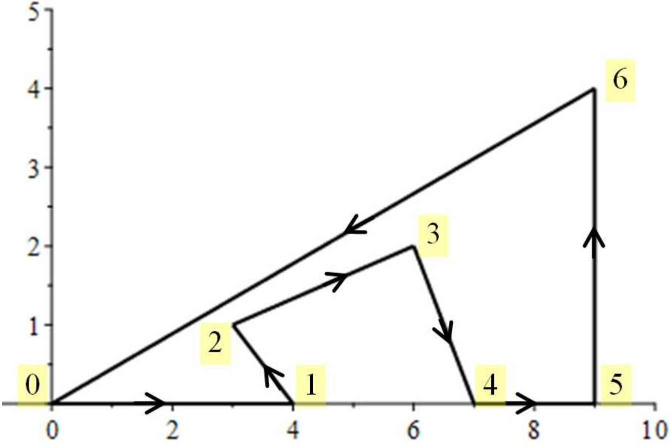
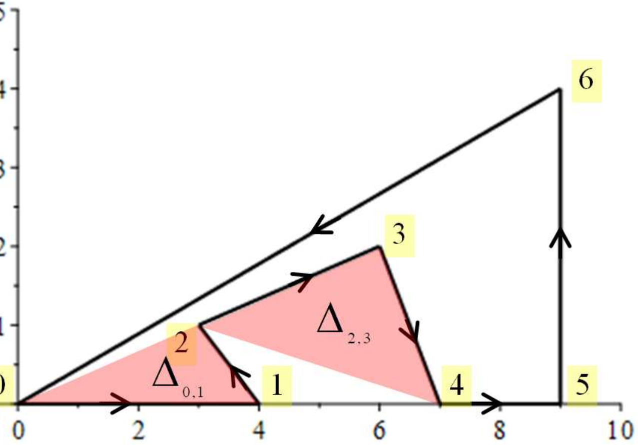
Triangles in polygons

Let P be a simple polygon on n vertices, $0, 1, \ldots, n-1$, oriented counterclockwise. We are interested in the areas of the triangles formed by joining vertices of P to 'opposite edges'.



Signed areas

Denote by Δ_{ij} the area of the triangle on polygon vertices $i, j, j+1$, the numbering taken modulo n. This area is taken as positive or negative according to whether $i, j, j+1, i$ has counterclockwise or clockwise orientation relative to the orientation of the polygon.



Left we have highlighted areas $\Delta_{0,1} = 2$ and $\Delta_{2,3} = -7/2$.

Note also $\Delta_{0,2} = \Delta_{0,4} = 0$.

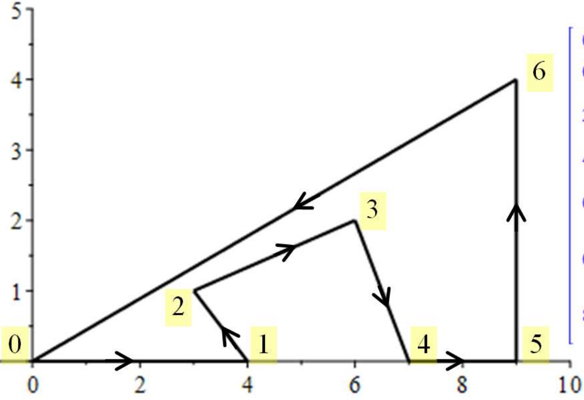
The Delta matrix

For an n-vertex polygon the Δ_{ij} form an $n \times n$ matrix.
The main and 1st lower diagonals are zero:

$$\Delta_{i,i} = 0, \Delta_{i+1,i} = 0.$$

The 1st upper diagonal equals the 2nd lower diagonal:

$$\Delta_{i,i+1} = \Delta_{i+2,i}.$$



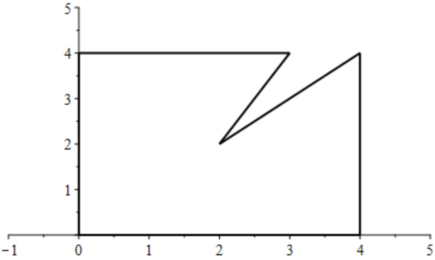
$$\begin{bmatrix} 0 & 2 & 0 & -7 & 0 & 18 & 0 \\ 0 & 0 & -2 & -3 & 0 & 10 & 8 \\ 2 & 0 & 0 & -\frac{7}{2} & 1 & 12 & \frac{3}{2} \\ 4 & -2 & 0 & 0 & 2 & 6 & 3 \\ 0 & -\frac{3}{2} & -\frac{7}{2} & 0 & 0 & 4 & 14 \\ 0 & -\frac{5}{2} & -\frac{9}{2} & 2 & 0 & 0 & 18 \\ 8 & -\frac{9}{2} & \frac{3}{2} & 4 & 4 & 0 & 0 \end{bmatrix}$$

Properties of the Delta matrix 1: Rank

0	−2	2	−4	8	$\frac{3}{2}$	12	$\frac{9}{2}$	$\frac{13}{2}$	8	0	$\frac{21}{2}$	$-\frac{7}{2}$	2	−2	0
0	0	2	−2	$\frac{11}{2}$	$\frac{3}{2}$	9	3	6	$\frac{11}{2}$	$\frac{3}{2}$	8	−1	$\frac{5}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
−2	0	0	−2	4	$-\frac{1}{2}$	8	$\frac{1}{2}$	$\frac{17}{2}$	4	4	$\frac{17}{2}$	$\frac{1}{2}$	4	2	4
−2	2	0	0	$\frac{3}{2}$	$-\frac{1}{2}$	5	−1	8	$\frac{3}{2}$	$\frac{11}{2}$	6	3	$\frac{9}{2}$	$\frac{9}{2}$	$\frac{11}{2}$
−4	2	−2	0	0	$-\frac{5}{2}$	4	$-\frac{7}{2}$	$\frac{21}{2}$	0	8	$\frac{13}{2}$	$\frac{9}{2}$	6	6	8
$-\frac{3}{2}$	$\frac{7}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	0	0	3	$-\frac{3}{2}$	7	0	6	4	$\frac{9}{2}$	$\frac{9}{2}$	6	6
$-\frac{3}{2}$	$\frac{11}{2}$	$\frac{1}{2}$	$\frac{7}{2}$	$-\frac{5}{2}$	0	0	−3	$\frac{13}{2}$	$-\frac{5}{2}$	$\frac{15}{2}$	$\frac{3}{2}$	7	5	$\frac{17}{2}$	$\frac{15}{2}$
$\frac{3}{2}$	$\frac{13}{2}$	$\frac{7}{2}$	$\frac{9}{2}$	$-\frac{3}{2}$	3	0	0	$\frac{5}{2}$	$-\frac{3}{2}$	$\frac{9}{2}$	$-\frac{1}{2}$	6	3	$\frac{15}{2}$	$\frac{9}{2}$
3	9	5	7	$-\frac{7}{2}$	$\frac{9}{2}$	−3	0	0	$-\frac{7}{2}$	$\frac{9}{2}$	−4	8	$\frac{5}{2}$	$\frac{19}{2}$	$\frac{9}{2}$
$\frac{7}{2}$	$\frac{13}{2}$	$\frac{11}{2}$	$\frac{9}{2}$	0	5	1	$\frac{5}{2}$	0	0	2	−1	$\frac{9}{2}$	$\frac{3}{2}$	6	2

Easy properties of $\Delta(P)$ the matrix of $\Delta_{i,j}$s

1. The diagonal and 1st lower diagonal entries are zero.
2. The first upper diagonal is identical to the second lower diagonal, the same triangle areas being subtended from opposite directions. That is, for $i = 0, 1, \ldots, n-1$, $\Delta_{i,i+1} = \Delta_{i+2,i}$.
3. In row i the first and second upper diagonal entries sum to same as the first and second lower diagonal entries in row $i+3$. That is $\Delta_{i,i+1} + \Delta_{i,i+2} = \Delta_{i+3,i} + \Delta_{i+3,i+1}$.



$$\begin{bmatrix} 0 & 8 & 0 & 1 & 6 & 0 \\ 0 & 0 & 4 & -3 & 6 & 8 \\ 8 & 0 & 0 & -1 & 0 & 8 \\ 4 & 4 & 0 & 0 & 3 & 4 \\ 8 & 2 & -1 & 0 & 0 & 6 \\ 8 & 8 & -4 & 3 & 0 & 0 \end{bmatrix}$$

Linear combinations of rows of $\Delta(P)$

A 3×3 diagonal block of $\Delta(P)$ has the following form

$$\begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ x & 0 & 0 \end{bmatrix}$$

The Easy Properties extend this to the row immediately beneath:

$$\begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ x & 0 & 0 \\ x+y-z & z & 0 \end{bmatrix}$$

Suppose $x \neq 0$. Write $\alpha_1 = z/x, \alpha_2 = -y/x, \alpha_3 = 1 - \alpha_1 - \alpha_2$. Then this extension is the linear combination

$$\text{Row } 4 = \alpha_1 \times \text{Row } 1 + \alpha_2 \times \text{Row } 2 + \alpha_3 \times \text{Row } 3$$

Surprisingly (to me) this same linear combination extends to the remainder of this same row.

Less immediate property of $\Delta(P)$

The matrix $\Delta(P)$ is determined by its first three rows. Specifically, $\Delta(P)$ has rank 3.

$$\begin{bmatrix} 0 & 8 & 0 & 1 & 6 & 0 \\ 0 & 0 & 4 & -3 & 6 & 8 \\ 8 & 0 & 0 & -1 & 0 & 8 \\ 4 & 4 & 0 & 0 & 3 & 4 \\ 8 & 2 & -1 & 0 & 0 & 6 \\ 8 & 8 & -4 & 3 & 0 & 0 \end{bmatrix}$$

By the easy properties, a diagonal 3×3 block determines the 3 entries immediately below.

As we have seen, we can write these 3 entries explicitly as a linear combination of the rows above.

And this same linear combination applies across the whole row.

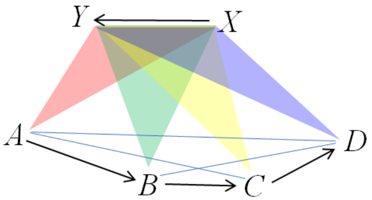
$$\text{Row 4} = \frac{z}{x} \times \text{Row 1} - \frac{y}{x} \times \text{Row 2} + \frac{x+y-z}{x} \times \text{Row 3},$$

provided $x \neq 0$.

A theorem about triangles

Suppose we have 6 points, A, B, C, D, X, Y, arranged in counterclockwise order in the plane. Using $|ABC|$, etc, to denote the area of the triangle on the points indicated, we have

$$|ABC| \times |XYD| + |ACD| \times |XYB| = |BCD| \times |XYA| + |ABD| \times |XYC|.$$



In the special case where the four triangles on base XY all had unit area this would just equate two ways of expressing the quadrilateral area $|ABCD|$. In fact it remains true in general.

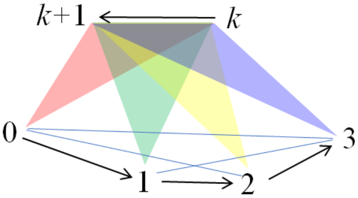
Applying the triangles theorem to the $\Delta_{i,j}$s

Without loss of generality take A, B, C, D to be the first four vertices, $0, 1, 2, 3$, of polygon P. The triangle theorem becomes

$$\Delta_{0,1} \times |XY3| + \Delta_{0,2} \times |XY1| = \Delta_{1,2} \times |XY0| + \Delta_{3,0} \times |XY2|.$$

Further, taking X, Y to be a subsequent edge $[k, k+1]$ of P, this becomes

$$\Delta_{0,1} \times \Delta_{3,k} + \Delta_{0,2} \times \Delta_{1,k} = \Delta_{1,2} \times \Delta_{0,k} + \Delta_{3,0} \times \Delta_{2,k}.$$



0	$\Delta_{0,1}$	$\Delta_{0,2}$	$\cdots$	$\Delta_{0,k}$	$\cdots$
0	0	$\Delta_{1,2}$	$\cdots$	$\Delta_{1,k}$	$\cdots$
$\Delta_{0,1}$	0	0	$\cdots$	$\Delta_{2,k}$	$\cdots$
$\Delta_{0,1} + \Delta_{0,2} - \Delta_{1,2}$	$\Delta_{1,2}$	0	$\cdots$	$\Delta_{3,k}$	$\cdots$

Rearranging, provided that $\Delta_{0,1} \neq 0$,

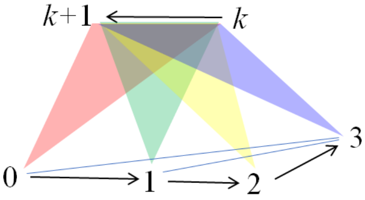
$$\Delta_{3,k} = \alpha_1 \Delta_{0,k} + \alpha_2 \Delta_{1,k} + \alpha_3 \Delta_{2,k},$$

with the α_i as on the earlier slide.

Dealing with zero-area $\Delta_{i,j}$s

If three consecutive polygon vertices are collinear then we have a zero area triangle.

E.g. here $\Delta_{0,1} = 0$ in the application of the triangle theorem, so we cannot write $\Delta_{3,k}$ in terms of the three previous rows of the matrix.



$$\Delta_{0,1} \times \Delta_{3,k} + \Delta_{0,2} \times \Delta_{1,k} = \Delta_{1,2} \times \Delta_{0,k} + \Delta_{3,0} \times \Delta_{2,k}.$$

In fact, these three previous rows are found to be already linearly dependent:

$$0 = -\Delta_{0,2} \times \Delta_{1,k} + \Delta_{1,2} \times \Delta_{0,k} + \Delta_{3,0} \times \Delta_{2,k}.$$

Properties of the Delta matrix 2: Eigenvectors

$$-\frac{1}{4} \quad \frac{3}{4} \quad \frac{1}{2} \quad -1 \quad 0 \Big] \times \begin{bmatrix} 0 & 8 & 0 & 1 \\ 0 & 0 & 4 & -3 \\ 8 & 0 & 0 & -1 \\ 4 & 4 & 0 & 0 \\ 8 & 2 & -1 & 0 \\ 8 & 8 & -4 & 3 \end{bmatrix}$$

Eigenvalues and eigenvectors

A complex number λ is an eigenvalue of square matrix X if, for some complex row vector v,

$$\begin{aligned} vX &= \lambda v & &v \text{ is a left eigenvector} \\ Xv^T &= \lambda v^T & &v^T \text{ is a right eigenvector} \end{aligned}$$

E.g. $X = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ -2 & -2 & -2 \end{pmatrix}$

Eigenvalues are 0 and $1 \pm 2i$.

Right eigenvectors are

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} (-1+i)/2 \\ (-2+i)/2 \\ 1 \end{pmatrix}, \begin{pmatrix} (-1-i)/2 \\ (-2-i)/2 \\ 1 \end{pmatrix}.$$

A right eigenvector of Δ

If all rows of a square matrix X have the same sum s then s is an eigenvalue of X and the all-ones column vector is a right eigenvector.

$$X \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} s \\ \vdots \\ s \end{pmatrix} = s \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

In the matrix $\Delta(P)$ for a polygon P, the i-th row sum is $\Delta_{i,i+1} + \ldots + \Delta_{i,i+n-2}$ (indexing modulo n). This sums a collection of triangles which partition the interior of P. Therefore all rows of $\Delta(P)$ sum to A_P, the area of the polygon.
(This is not quite obvious, unless P is convex, because some of the triangles may lie partially outside the boundary of P. That the overall sum is A_P is the content of the Shoelace Formula for polygon area.)

Thus A_p is an eigenvalue of $\Delta(P)$ with corresponding right eigenvector the all-ones vector.

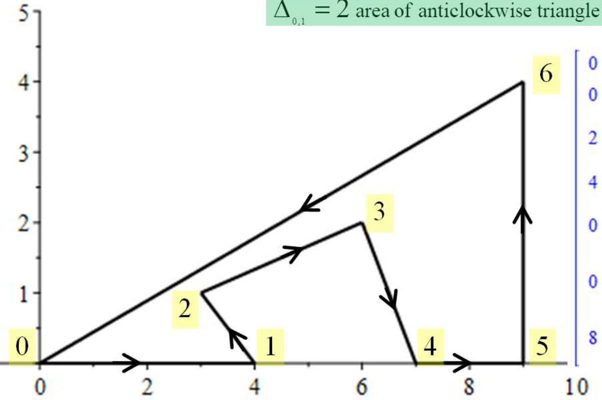
Delta matrix: all-ones right eigenvector

Our example polygon has area 13.

$\Delta_{0,5} = 18$ area of anticlockwise triangle 056

$\Delta_{0,3} = -7$ area of clockwise triangle 034

$\Delta_{0,1} = 2$ area of anticlockwise triangle 012



$$\begin{bmatrix} 0 & 2 & 0 & -7 & 0 & 18 & 0 \\ 0 & 0 & -2 & -3 & 0 & 10 & 8 \\ 2 & 0 & 0 & -\frac{7}{2} & 1 & 12 & \frac{3}{2} \\ 4 & -2 & 0 & 0 & 2 & 6 & 3 \\ 0 & -\frac{3}{2} & -\frac{7}{2} & 0 & 0 & 4 & 14 \\ 0 & -\frac{5}{2} & -\frac{9}{2} & 2 & 0 & 0 & 18 \\ 8 & -\frac{9}{2} & \frac{3}{2} & 4 & 4 & 0 & 0 \end{bmatrix}$$

Some left eigenvectors of Δ

Suppose rows $i, \ldots i + k$ of a square matrix X are linearly dependent. This is the same as saying that summing these rows with the appropriate weights, say, $\alpha_1, \ldots \alpha_k$, will give the zero vector.

Then multiplying X on the left by the row vector $v = (0, \ldots, 0, \alpha_1, \ldots, \alpha_k, 0, \ldots, 0)$ will give a row of zeros. That is, $vX = 0 \times v$, so v is a left eigenvector corresponding to an eigenvalue of zero.

We have seen that $\Delta(P)$, for an n-vertex polygon P, has $n - 3$ such dependencies. So zero is an eigenvalue of Δ of multiplicity $n - 3$. We can construct the corresponding left eigenvectors as shown earlier, making $n - 3$ applications of the triangle theorem.

Delta matrix: zero left eigenvector

One of three zero left eigenvectors:

$\alpha_1 = (-1)/4$

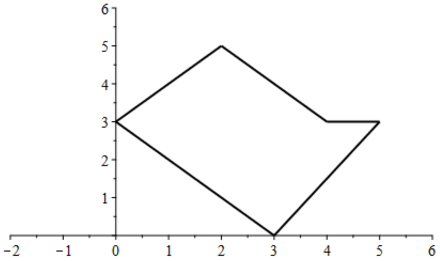
$\alpha_2 = -(-3)/4$

$\alpha_3 = 1-(-1/4)-(3/4)$

$$\begin{bmatrix} 0 & -\frac{1}{4} & \frac{3}{4} & \frac{1}{2} & -1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 8 & 0 & 1 & 6 & 0 \\ 0 & 0 & 4 & -3 & 6 & 8 \\ 8 & 0 & 0 & -1 & 0 & 8 \\ 4 & 4 & 0 & 0 & 3 & 4 \\ 8 & 2 & -1 & 0 & 0 & 6 \\ 8 & 8 & -4 & 3 & 0 & 0 \end{bmatrix}$$

(What are the corresponding right eigenvectors?)

Properties of the Delta matrix 3: Characteristic equation



$$sm := \begin{bmatrix} 0 & \frac{15}{2} & 0 & 4 & 0 \\ 0 & 0 & \frac{3}{2} & 4 & 6 \\ \frac{15}{2} & 0 & 0 & -1 & 5 \\ 6 & \frac{3}{2} & 0 & 0 & 4 \\ 6 & \frac{13}{2} & -1 & 0 & 0 \end{bmatrix}$$

$$csm := q^5 - 64\,q^3 - \frac{6279}{8}\,q^2$$

The characteristic polynomial of Δ

The eigenvalues of an $n \times n$ matrix X are precisely the roots of the degree-n polynomial in λ defined as $c(X, \lambda) = \det(\lambda I - X)$. This is the **characteristic polynomial** of X.

We will use q instead of λ when talking about Δ to distinguish it from the general case. From the previous slides we know we can almost write down $c(\Delta(P), q)$ explicitly, factorised in terms of its roots:

$$c(\Delta(P), q) = q^{n-3}(q - A_P)(q^2 + aq + b),$$

where A_P is the area of polygon P. The final factor is a quadratic in q which is our next objective!

Some bits of theory

The characteristic polynomial of an $n \times n$ matrix X is

$$c(X,\lambda) = \det(\lambda I - X) = \begin{pmatrix} \lambda - x_{0,0} & -x_{0,1} & \cdots & -x_{0,n-1} \\ -x_{1,0} & \lambda - x_{1,1} & \cdots & \\ & & \ddots & \\ -x_{n-1,0} & & \cdots & \lambda - x_{n-1,n-1} \end{pmatrix}.$$

It has the form

$$c(X,\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_n,$$

where

1. $a_0 = 1$ (i.e. c is monic);
2. $a_1 = -(x_{0,0} + x_{1,1} + \ldots + x_{n-1,n-1}) = -\mathrm{Tr}(X)$;
3. $a_2 =$ sum of all 2×2 principal minors of X;
4. $a_n = (-1)^n \det(X)$.

Our quadratic: the q term

We have (writing Δ and A for $\Delta(P)$ and A_P, respectively):

$$c(\Delta, q) = q^{n-3}(q - A)(q^2 + aq + b).$$

The q^{n-1} term is $-Aq^{n-1} + aq^{n-1}$.

This is equal to $-\mathrm{Tr}(\Delta) = 0$ (since $\Delta_{i,i} = 0$ for all i).

So we have $a = A$ and

$$c(\Delta, q) = q^{n-3}(q - A)(q^2 + Aq + b).$$

We would to know the value of b. In fact, we would like to specify the value of b *geometrically*.

Our quadratic: the constant term b

In
$$c(\Delta, q) = q^{n-3}(q - A)(q^2 + Aq + b),$$
The coefficient of q^{n-2} is $b - A^2$. The theory tells us this is the sum of all 2×2 principal minors of Δ.

$\Delta_{0,1} \times \Delta_{1,0}$

$\Delta_{1,3} \times \Delta_{3,1}$

$\Delta_{1,5} \times \Delta_{5,1}$

$\Delta_{3,5} \times \Delta_{5,3}$

$$\begin{bmatrix} 0 & 8 & 0 & 1 & 6 & 0 \\ 0 & 0 & 4 & -3 & 6 & 8 \\ 8 & 0 & 0 & -1 & 0 & 8 \\ 4 & 4 & 0 & 0 & 3 & 4 \\ 8 & 2 & -1 & 0 & 0 & 6 \\ 8 & 8 & -4 & 3 & 0 & 0 \end{bmatrix}$$

Because the diagonal is all zeros, each 2×2 principal minor is $-1\times$ the product of a row entry with the diagonally opposite column entry.

Our quadratic: completely specified?

Summing the 2×2 principal minors along a row gives precisely $-1\times$ the corresponding diagonal entry of Δ^2. But each principal minor product will appear twice in the calculation of the whole diagonal of Δ^2.

$$\begin{bmatrix} 0 & 8 & 0 & 1 & 6 & 0 \\ 0 & 0 & 4 & -3 & 6 & 8 \\ 8 & 0 & 0 & -1 & 0 & 8 \\ 4 & 4 & 0 & 0 & 3 & 4 \\ 8 & 2 & -1 & 0 & 0 & 6 \\ 8 & 8 & -4 & 3 & 0 & 0 \end{bmatrix}$$

$\Delta_{1,0} \times \Delta_{0,1}$

$\Delta_{1,2} \times \Delta_{2,1}$

$\Delta_{1,3} \times \Delta_{3,1}$

$\Delta_{1,4} \times \Delta_{4,1}$

$\Delta_{1,5} \times \Delta_{5,1}$

The coefficient of q^{n-2} in $c(\Delta, q)$ is therefore $-\frac{1}{2}\mathrm{Tr}(\Delta^2)$. That is $b - A^2 = -\frac{1}{2}\mathrm{Tr}(\Delta^2)$, and we have

$$c(\Delta, q) = q^{n-3}(q - A)\left(q^2 + Aq + A^2 - \frac{1}{2}\mathrm{Tr}(\Delta^2)\right).$$

Eigenvalues completely specified?

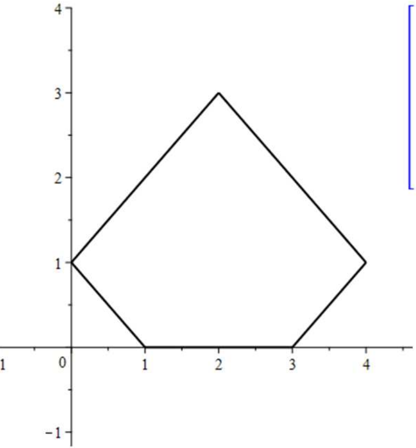
From $c(\Delta, q) = q^{n-3}(q - A)\left(q^2 + Aq + A^2 - \frac{1}{2}\mathrm{Tr}(\Delta^2)\right)$, The eigenvalues of Δ may be directly calculated as

$$\overbrace{0, \ldots, 0}^{n-3}, A, -\frac{A}{2} \pm \sqrt{2\mathrm{Tr}(\Delta^2) - 3A^2}.$$

However, we don't have the eigenvectors corresponding to the last two eigenvalues.

What is worse, unlike the first $n-2$ eigenvalues we have no geometric interpretation of the final two. What does it mean *geometrically* to sum products of pairs of triangle areas? Or to take the square of a matrix of triangle areas?

An example



$$\begin{bmatrix} 0 & 1 & 2 & 4 & 0 \\ 0 & 0 & 1 & 4 & 2 \\ 1 & 0 & 0 & 2 & 4 \\ 2 & 1 & 0 & 0 & 4 \\ 2 & 3 & 2 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 10 & 4 & 1 & 8 & 26 \\ 13 & 10 & 4 & 2 & 20 \\ 12 & 15 & 10 & 4 & 8 \\ 8 & 14 & 13 & 12 & 2 \\ 2 & 2 & 7 & 24 & 14 \end{bmatrix}$$

$$\Delta \qquad \Delta^2$$

$$c(\Delta, q) \quad q^5 - 28\,q^3 - 147\,q^2$$

$$c(\Delta, q)/q^2(q - A) \quad q^2 + 7\,q + 21$$

$$0,\ 0,\ 7,\ -\frac{7}{2} - \frac{I\sqrt{35}}{2},\ -\frac{7}{2} + \frac{I\sqrt{35}}{2}$$

What is the $\sqrt{35}$ telling us about the polygon??

Thank you for sharing my frustration!