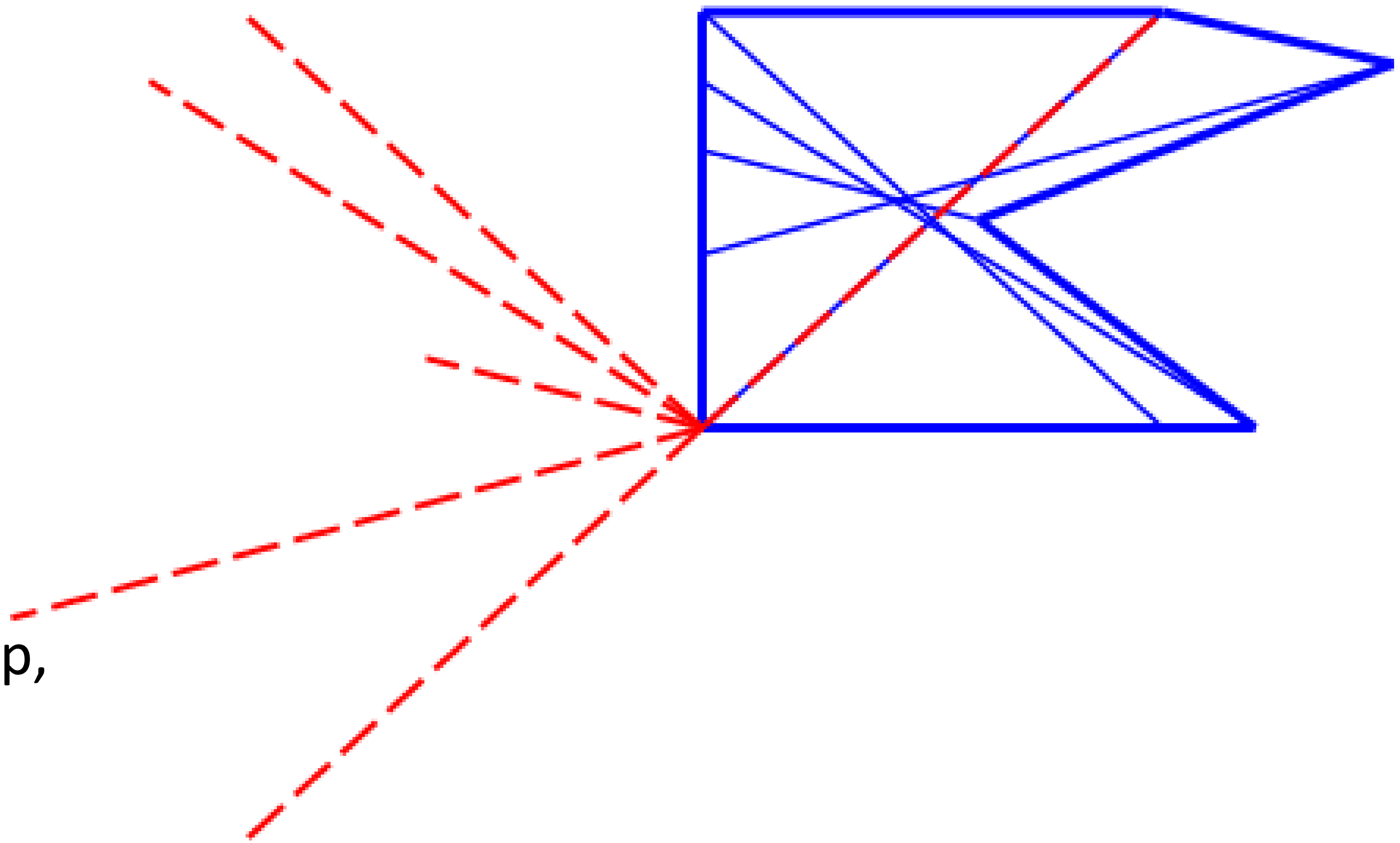
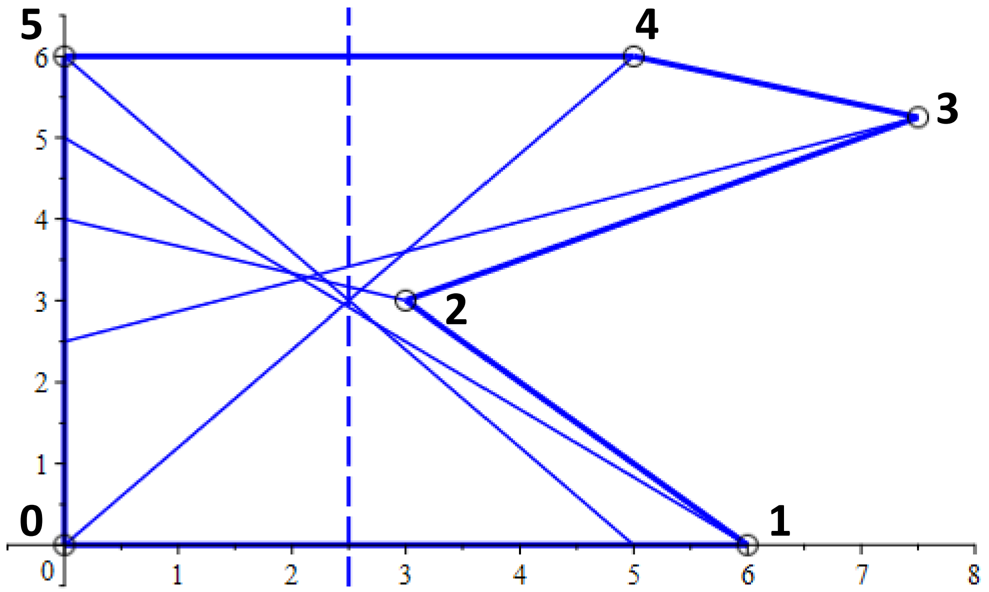
All-angles polygon bisection



Robin Whitty
Maths Study Group,
March 27, 2026

Bisection-convexity



Bisection-convex: every area-bisecting straight line meets the polygon in exactly two points (or one point and one edge – non-strict bisection-convexity)

A. Berele, S. Catoiu / Advances in Applied Mathematics 137 (2022) 102342

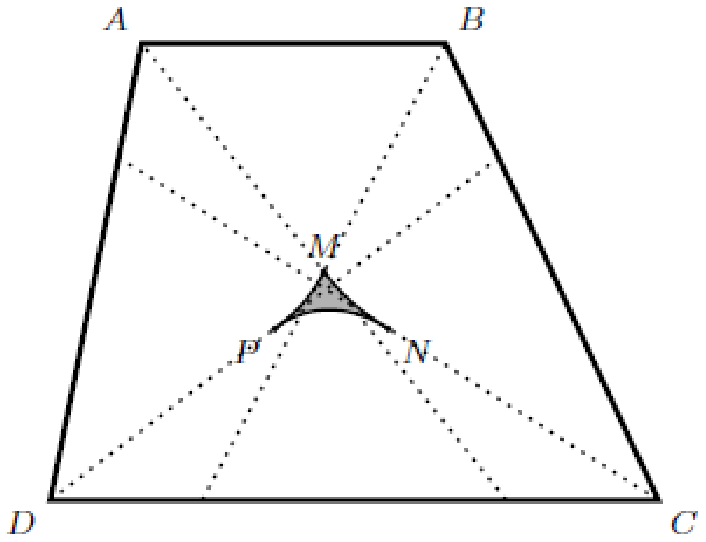


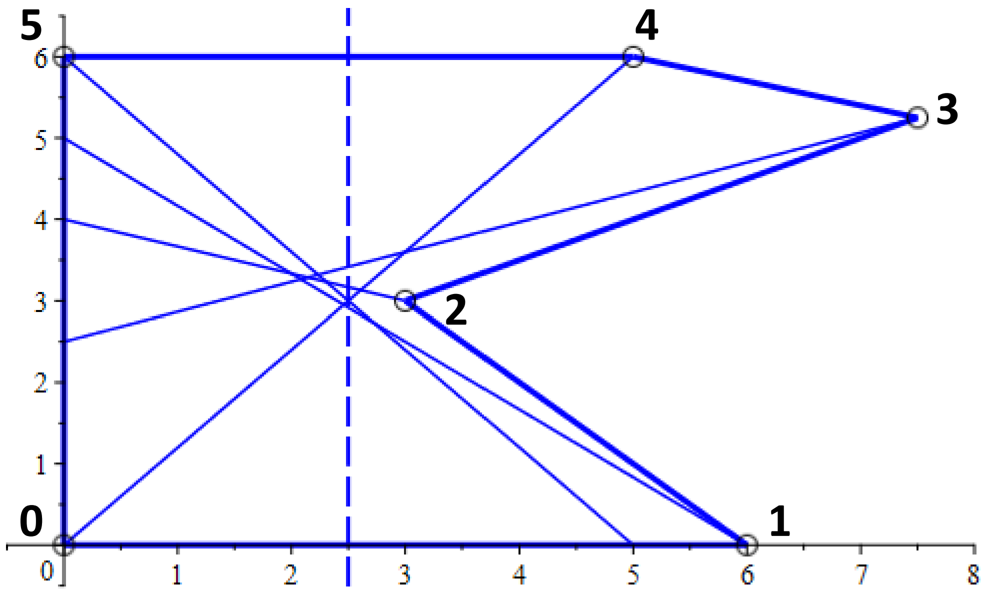
Fig. 0.7. The area-bisecting envelope of a trapezoid.

Theorem (Fechtor-Pradines) A bisection-convex polygon has an envelope, formed from sections of hyperbolae, to which every bisecting line is tangent.

Fechtor-Pradines, N., “Bisection envelopes”, *Involve,* Vol. 8, No. 2, 2015, pp. 307–328.
Berele, A. and Catoiu, S., “Bisecting envelopes of convex polygons”, *Advances in Applied Mathematics*, Vol. 137, 2022, article no. 102342.

Bisection-convexity

Theorem: A polygon is bisection-convex if and only if, for each vertex, the bisecting straight line through that vertex meets the polygon in two points (or one point and one line).



The bisecting line from a vertex to an opposite edge will be called a **bisecting chord**.

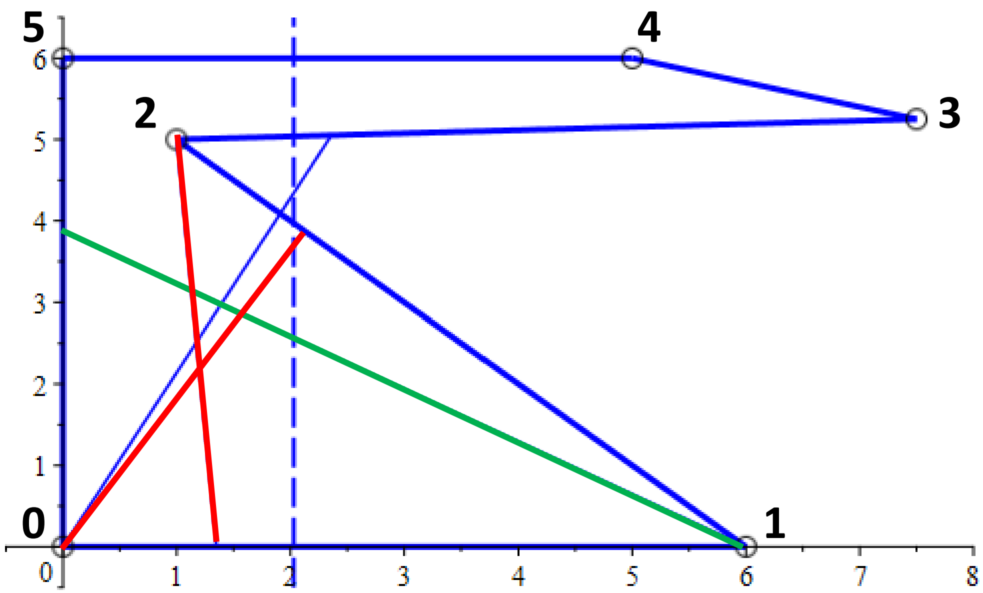
Any bisecting line through a bisection-convex polygon is a rotation of some bisecting chord about a suitable point on that chord (we are rotating the chord to become tangent to the bisecting envelope).

This means we can specify a bisecting line for **any given angle**.

E.g. the vertical dashed line is bisecting. It is the rotation of the bisecting chord through vertex 0 (and vertex 4) about its midpoint.

Non-bisection-convexity

Here is the same polygon but with vertex 2 translated along the straight line joining vertices 1 and 5. The vertical dashed line is again bisecting but meets the polygon in four points. So this is no longer a bisection-convex polygon.



It may still have some bisecting chords. E.g. the green chord (vertex 1 to edge [5, 0]) is bisecting and meets the polygon in two points.

The red chords (from vertices 0 and 2) are bisecting in a 'local' sense: half the polygon area lies to their right and half to its left.
More precisely: follow the chords from vertex to opposite edge; return to the start vertex clockwise around the polygon edges. A region having half the polygon area will have been described.

But we can no longer specify all-angles bisection by rotating a fixed set of chords (no bisecting envelope).

Shermer's algorithm

Thomas Shermer gave an elegant and efficient (linear in number of polygon vertices) algorithm for bisecting any simple polygon.

1. Rotate the polygon so that the desired angle of bisection is vertical;
2. Trapezoidize the polygon vertically;
3. Specify a quadratic equation in terms of the areas of the trapezoids;
4. The appropriate solution to the quadratic will be the x-coordinate of the bisection.

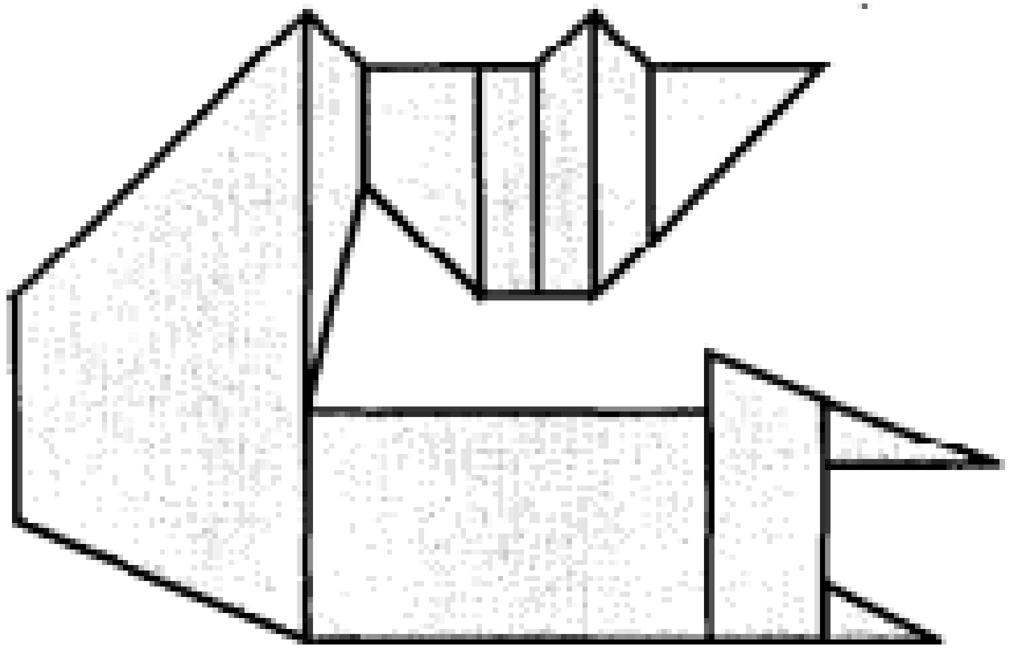


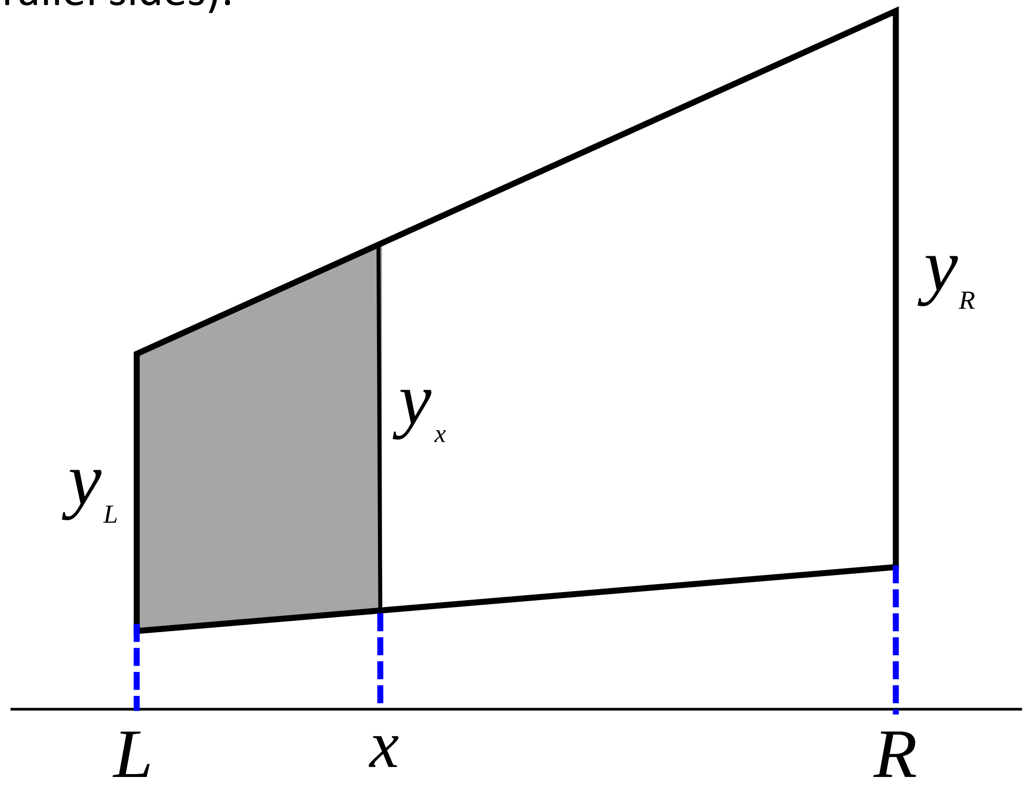
Fig. 2. A trapezoidized polygon.

Shermer, T.C., "A linear algorithm for bisecting a polygon", *Information Processing Letters*, Vol. 41, Issue 3, 1992, pp. 135-140.

Bisecting trapezoids (as used by Shermer)

Take any vertical trapezoid (quadrilateral with two vertical parallel sides):

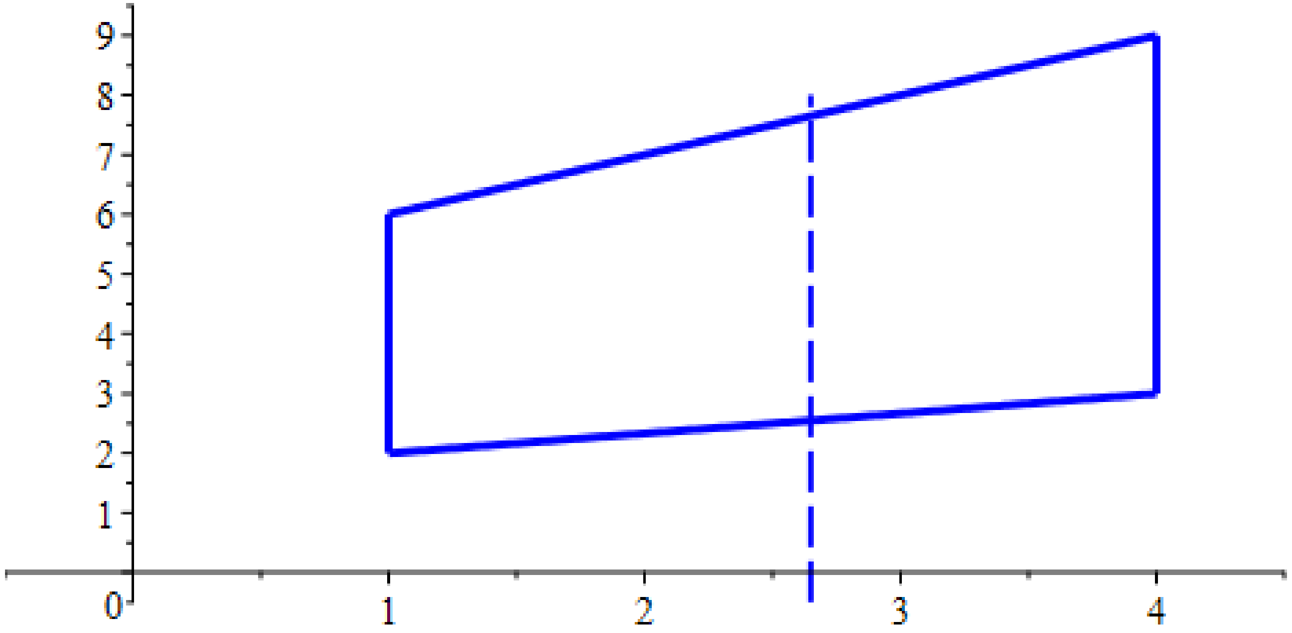
$$\text{Shaded area} = (x-L)\frac{(Y_L+Y_x)}{2}$$
$$= Ax^2 + 2Bx + C, \text{ where}$$
$$A = \frac{(Y_R - Y_L)}{2(R-L)}$$
$$B = -AL + \frac{Y_L}{2}$$
$$C = AL^2 - LY_L$$



So solving $Ax^2 + 2Bx + C = \frac{1}{2} \times \text{trapezoid area}$

will give the bisecting value of x

Bisecting trapezoids example



$$Y_R = 6,$$
$$Y_L = 4,$$
$$R = 4,$$
$$L = 1$$

$$\text{trapezoid area} = (R-L)\frac{Y_L + Y_R}{2} = 15$$

$$A = \frac{Y_R - Y_L}{2(R-L)} = \frac{1}{3},$$

$$B = -AL + \frac{Y_L}{2} = \frac{5}{3},$$

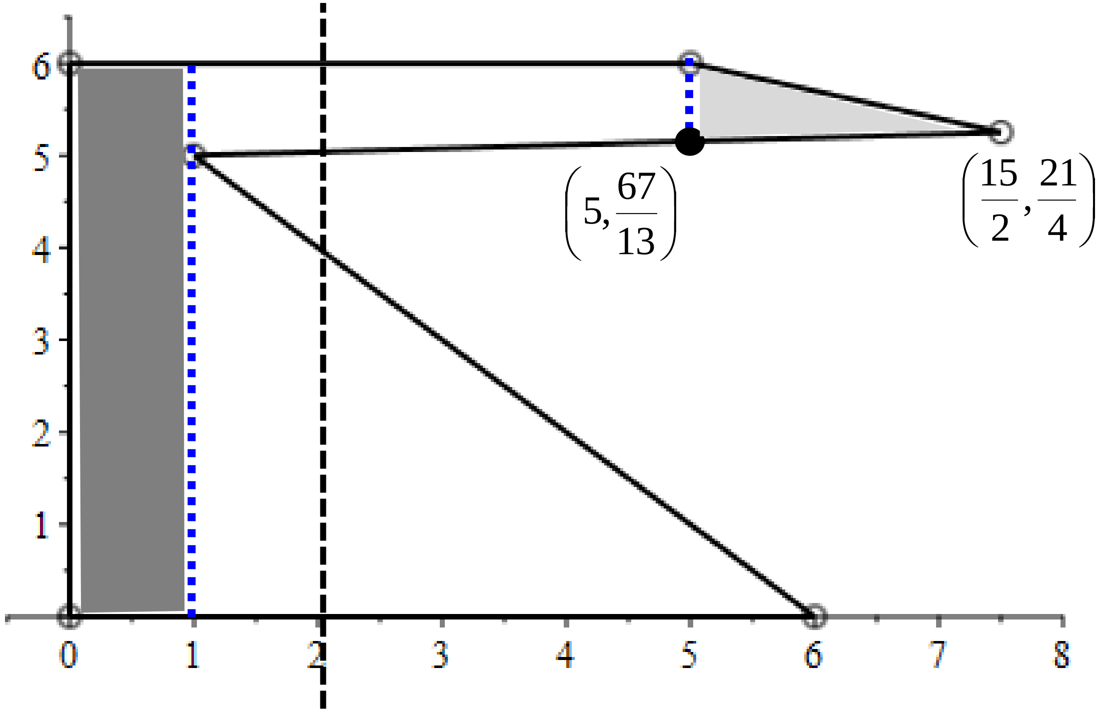
$$\mathrm{C} = AL^2 - LY_L = -\frac{11}{3}$$

And now solving $\frac{1}{3}x^2 + \frac{10}{3}x - \frac{11}{3} - \frac{15}{2} = 0$ gives

$$x = -5 \pm \frac{3\sqrt{26}}{2} \approx -5 \pm 7.65$$

Shermer applied to our non-bisection-convex polygon

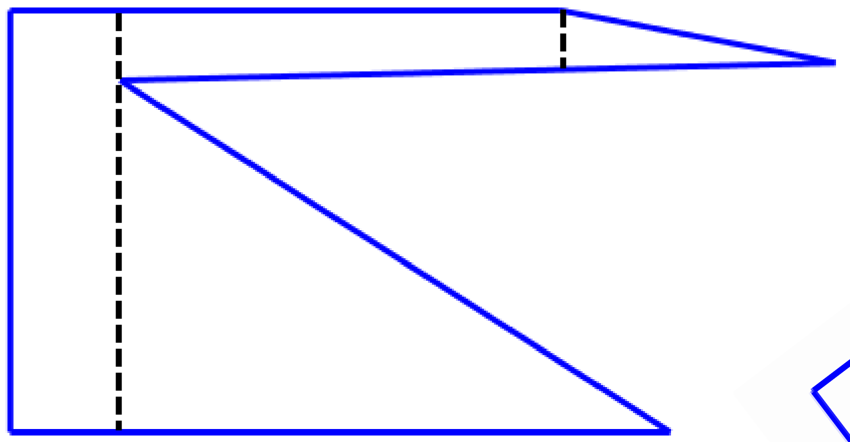
Shermer’s algorithm starts by taking vertical lines from each polygon vertex to divide the polygon into trapezoids (sometimes rectangular or triangular of course).

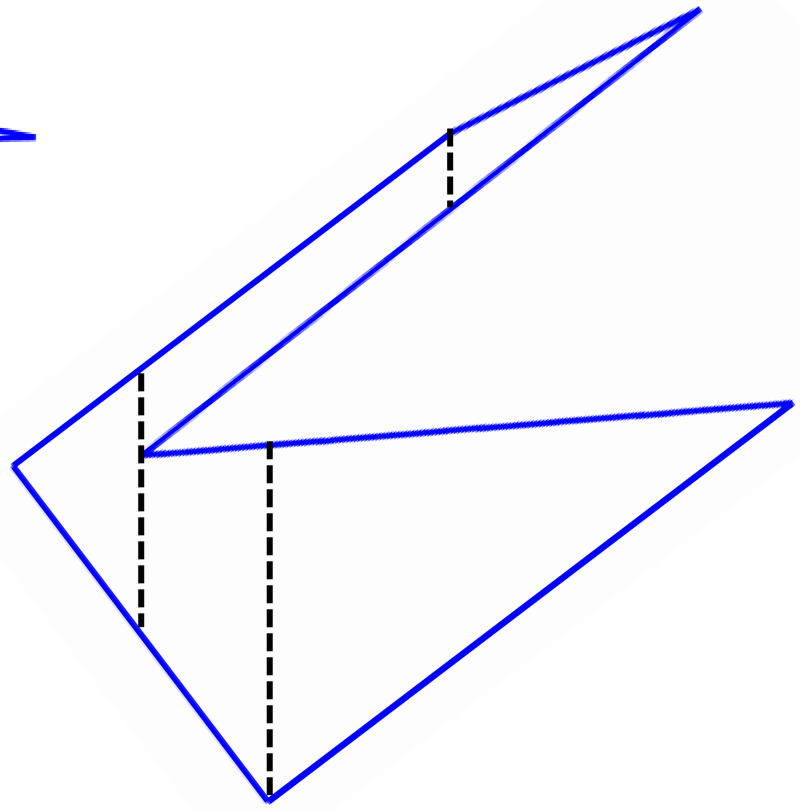


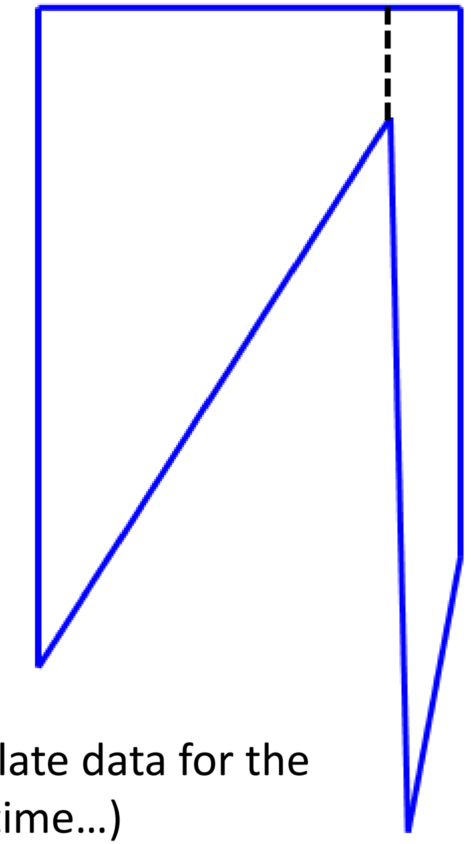
1. Determine (easy) which trapezoids are to the left (dark shaded) or right (light shaded) of the bisecting vertical line and which are split by it (no shading).
2. The bisecting quadratic equations for the split trapezoids may be summed to give a single quadratic.
3. Solve the joint quadratic to give the x coordinate of the bisection.

Shermer’s algorithm for many angles

Shermer’s algorithm depends on vertical bisection of vertical trapezoids (having vertical parallel sides). Changing to a non-vertical bisection requires a rotation and re-trapezoidization of the polygon.



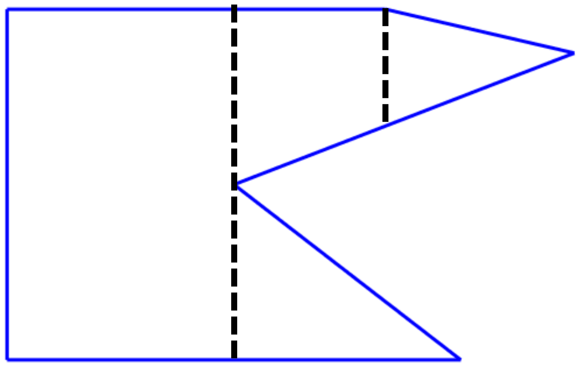


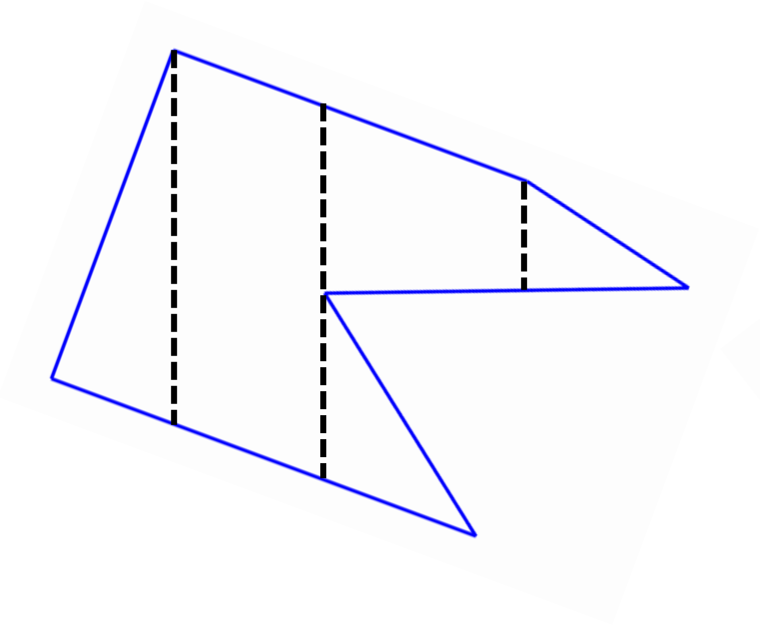


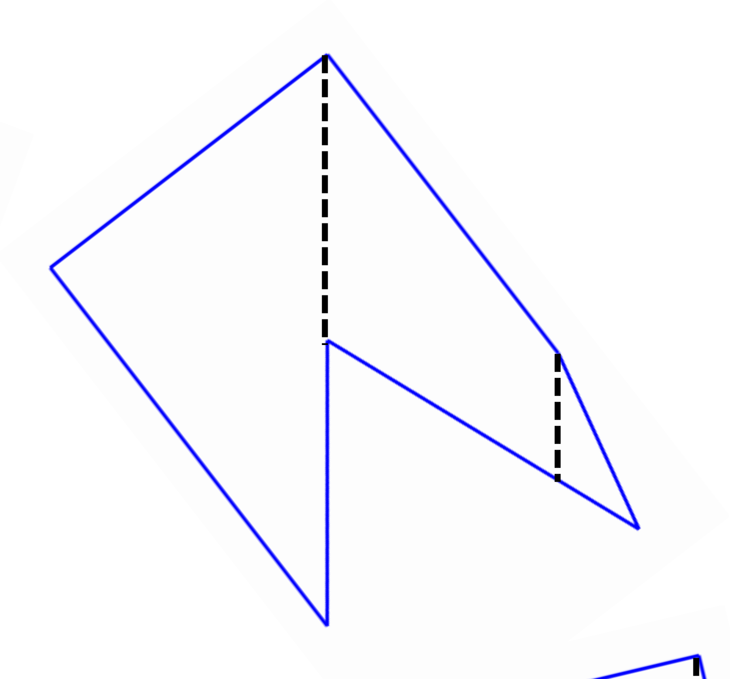
This may be inevitable for non-bisection-convex polygons. But for bisection-convex we can tabulate data for the polygon which can be used for any new bisection angle (but maybe the tabulation is not linear time…)

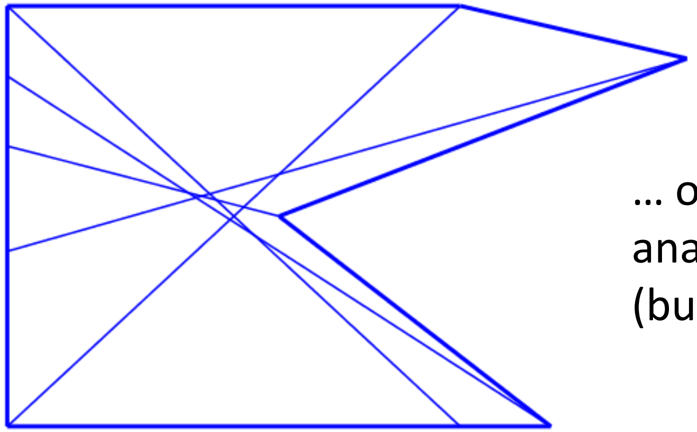
All-angles bisection

Choice: accept repeated trapezoidization….

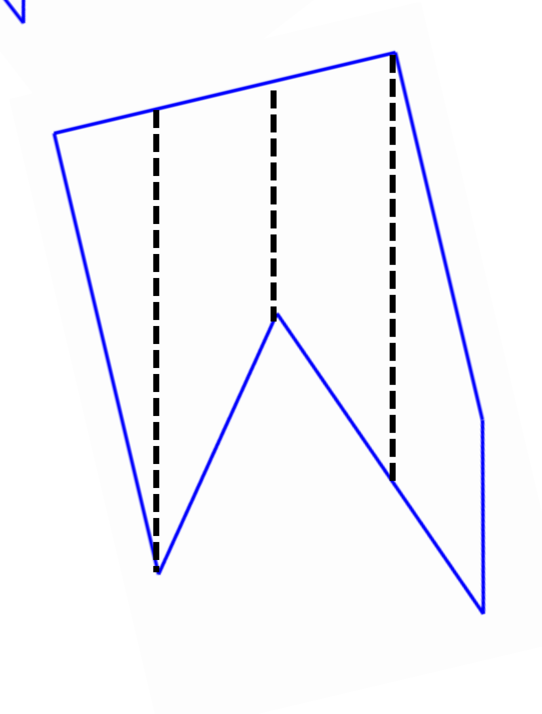






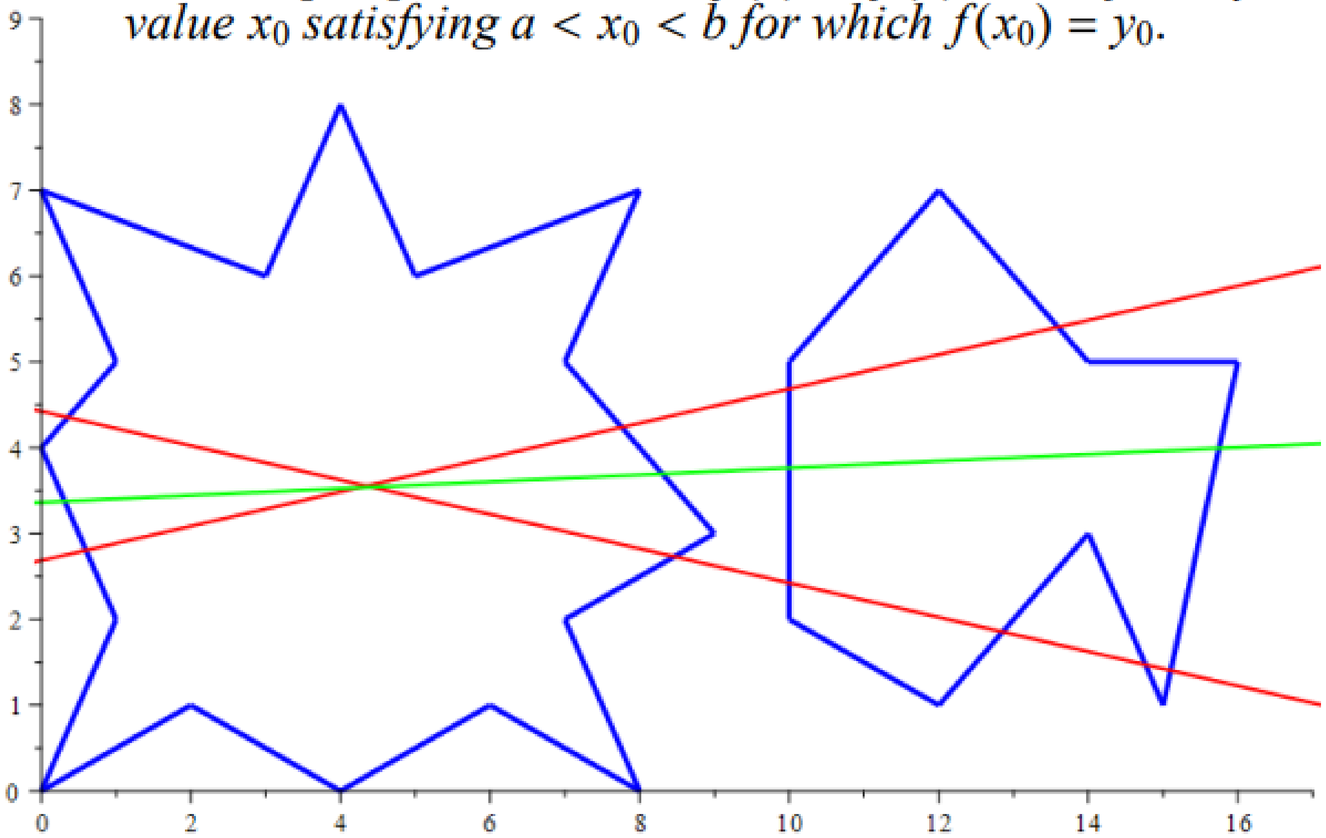


… or do a (possibly more time-consuming) pre-analysis to allow bisection with rapid angle change (but only for bisection-convex polygons).



On theoremoftheday.org

The Intermediate Value Theorem *Let $f(x)$ be a real-valued function which is continuous on the closed interval $[a,b]$ and such that $f(a) < f(b)$. Then for any value y_0 satisfying $f(a) < y_0 < f(b)$, there is a value x_0 satisfying $a < x_0 < b$ for which $f(x_0) = y_0$.*



In our illustration above, the three straight lines bisect the left-hand pancake. One crosses the right-hand pancake too low down to bisect its area and another crosses too high. But the middle straight line bisects and achieves the conclusion of the Two Pancakes Theorem. To be precise, however, this right-hand bisection is approximate. Indeed, the Intermediate Value Theorem itself is non-constructive, and in any particular application it may or may not be the case that a direct construction of an intermediate value is possible. Thus, there are known exact methods for bisecting certain types of polygon in a given direction and these are being applied above to the left-hand pancake. But exact bisection of two pancakes is not in general available: the intermediate value exists but must apparently be approximated.

The Pancake Theorem *If P is a simple closed curve in the plane, then for any specified angle there is a unique straight line at this angle to the horizontal which bisects the area of P.*

Proof. Let P have area A. Let the specified angle be given as θ to the horizontal. Let a straight line at angle θ lie entirely below P. Define $f(y)$ to be the function which records the area of that part of curve P which lies below the line when it is translated vertically by y. Then $f(0) = 0$, $f(h) = A$, for some sufficiently high value of h, and f is continuous on the interval $[0,h]$. So the Intermediate Value Theorem says there is height h_0 with $f(h_0) = A/2$. Moreover, f is a strictly increasing function and therefore there is a unique line at angle θ bisecting P_1.

The Two Pancakes Theorem *Let P_1 and P_2 be simple closed curves in the plane. Then there is a straight line in the plane which simultaneously bisects the area of both P_1 and P_2.*

Proof. We prove a version in which P_2 is entirely to the right of P_1 in the positive quadrant. Let the area of P_2 be A. Define $g(\theta)$ to be the function which records the area of that part of curve P_2 which lies below the unique straight line at angle θ to the horizontal which bisects the area of P_1. Then it is clear that $g(\theta_1) = 0$ and $g(\theta_2) = A$ for suitably chosen angles $-\tau/4 \le \theta_1$ and $\theta_2 \le \tau/4$. Then, subject to a proof that g is continuous on the interval $[\theta_1, \theta_2]$, the Intermediate Value Theorem confirms that there is a value θ_0 for which $g(\theta_0) = A/2$.

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The Pancake Theorem

Posted on July 1, 2022 by Ofir

We all know pancakes and how delicious they can be if prepared properly, and it is only natural wanting to share it with your closest friend. However, cutting a pancake exactly in half, one for you and one for your friend, is not trivial matter. It is even less trivial when you understand that your friend also has a favorite pancake which he wants to share as well, so now we need to double our cutting process. In this post we will show that with a bit of mathematics, not only you can cut both pancakes exactly in half, but you can do it simultaneously with a single straight line.

For the Hebrew listeners among you – here is a video I made on this subject: https://youtu.be/_xh6yQocHj4

And for the English listeners: https://www.youtube.com/watch?v=974OcQH-9og

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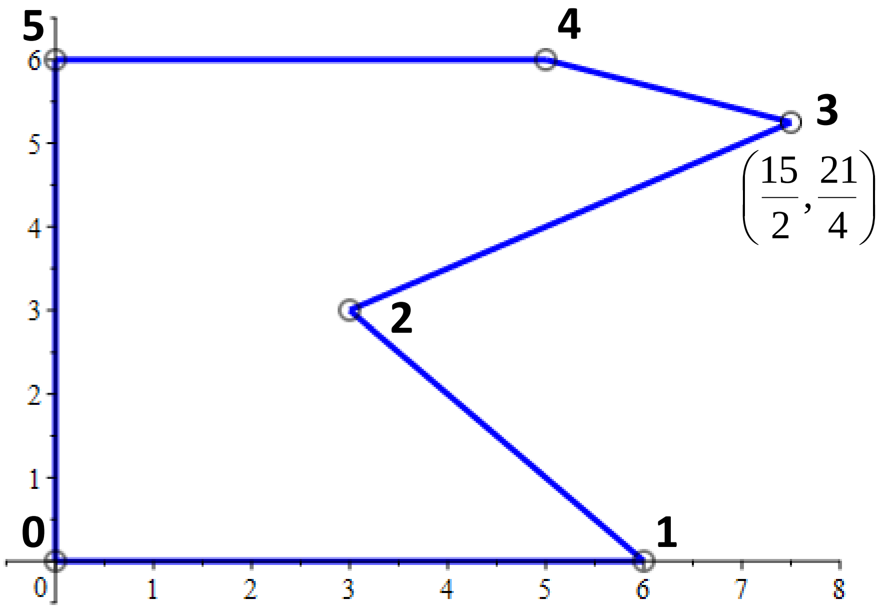
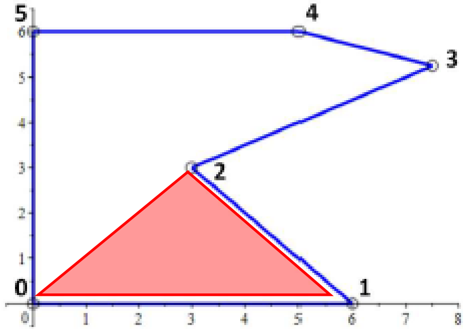
Bisection-convex pre-analysis I
Triangle areas matrix Δ_P

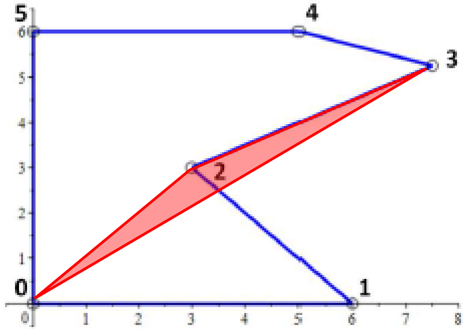


$$
\Delta_P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{bmatrix}
0 & 9 & -\frac{27}{8} & \frac{75}{8} & 15 & 0 \\
0 & 0 & -\frac{81}{8} & \frac{57}{8} & 15 & 18 \\
9 & 0 & 0 & \frac{9}{2} & \frac{15}{2} & 9 \\
\frac{63}{4} & -\frac{81}{8} & 0 & 0 & \frac{15}{8} & \frac{45}{2} \\
18 & -\frac{15}{2} & \frac{9}{2} & 0 & 0 & 15 \\
18 & 0 & \frac{81}{8} & \frac{15}{8} & 0 & 0
\end{bmatrix}
\end{array}
$$

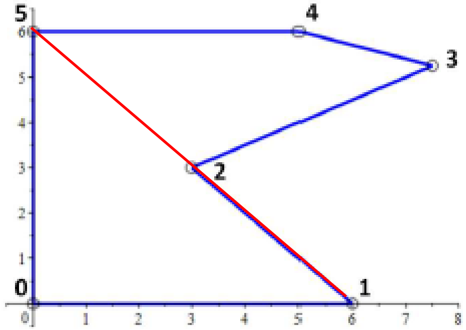
In matrix Δ_P entry $\Delta_{i,j}$ is the area of the triangle on vertex i and opposite edge $[j, j+1]$



$\Delta_{0,1} = 9$

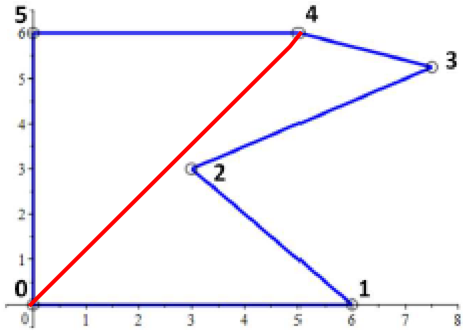


$\Delta_{0,2} = -27/8$

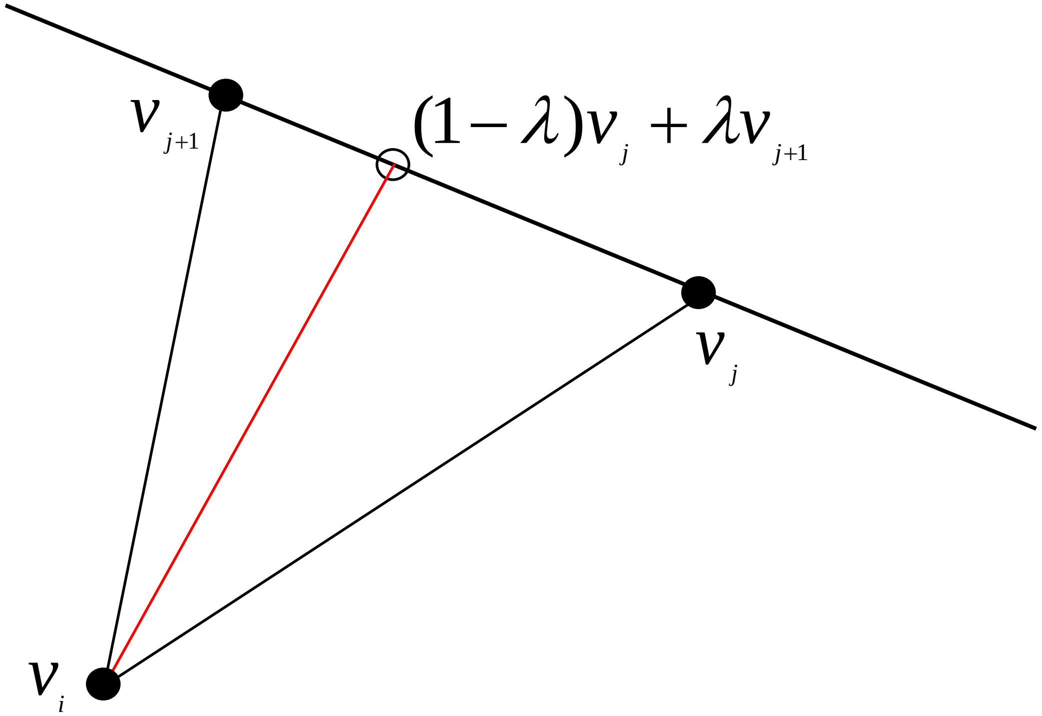


$\Delta_{5,2} = 0$

Bisection-convex pre-analysis II
Spotting bisecting chords



Polygon area is 30, so this triangle is half the area

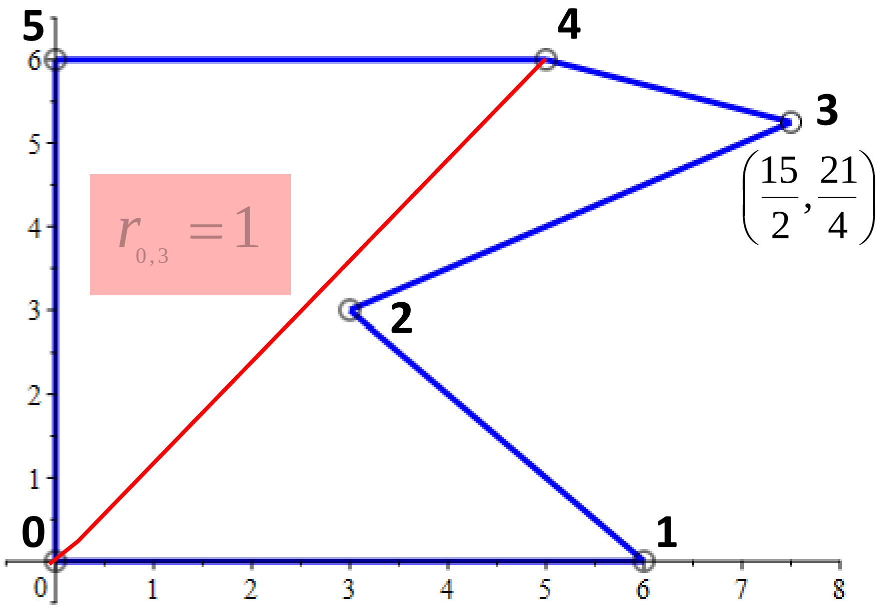


$\lambda = 0$: the red line goes to v_j

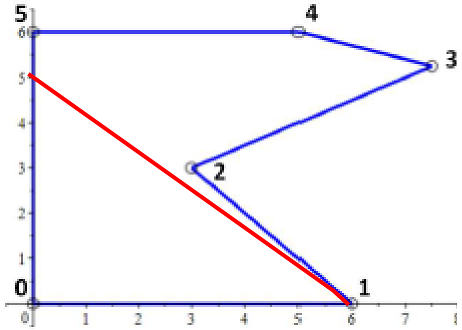
$\lambda = 1$: the red line goes to v_{j+1}

$\lambda \notin [0,1]$: red line goes to a point outside edge $[v_j, v_{j+1}]$

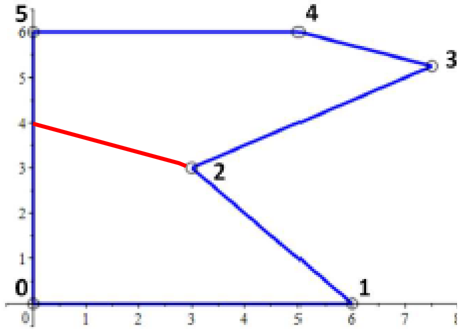
Bisection-convex pre-analysis III
Local bisections matrix $\boldsymbol{R_P}$



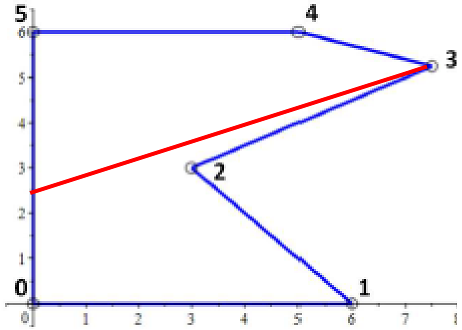
$$R_P = \begin{array}{c|c} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \left[\begin{array}{cccccc} 0 & \frac{5}{3} & -\frac{16}{9} & 1 & 0 & 0 \\ 0 & 0 & -\frac{40}{27} & \frac{67}{19} & \frac{6}{5} & \frac{1}{6} \\ -\frac{2}{3} & 0 & 0 & \frac{10}{3} & \frac{7}{5} & \frac{1}{3} \\ -\frac{25}{42} & \frac{67}{27} & 0 & 0 & 8 & \frac{7}{12} \\ 0 & \frac{12}{5} & -\frac{7}{3} & 0 & 0 & 1 \\ \frac{5}{6} & 0 & -\frac{8}{27} & -7 & 0 & 0 \end{array}\right] \end{array}$$



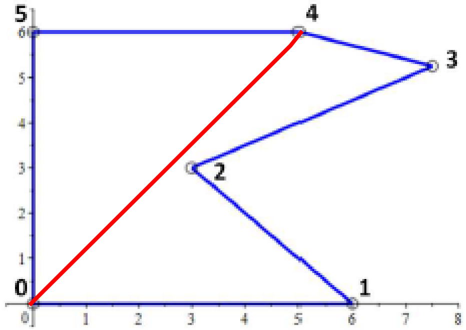
$r_{1,5} = 1/6$



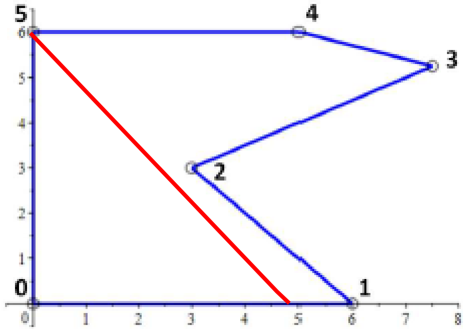
$r_{2,5} = 1/3$



$r_{3,5} = 7/12$



$r_{4,5} = 1$

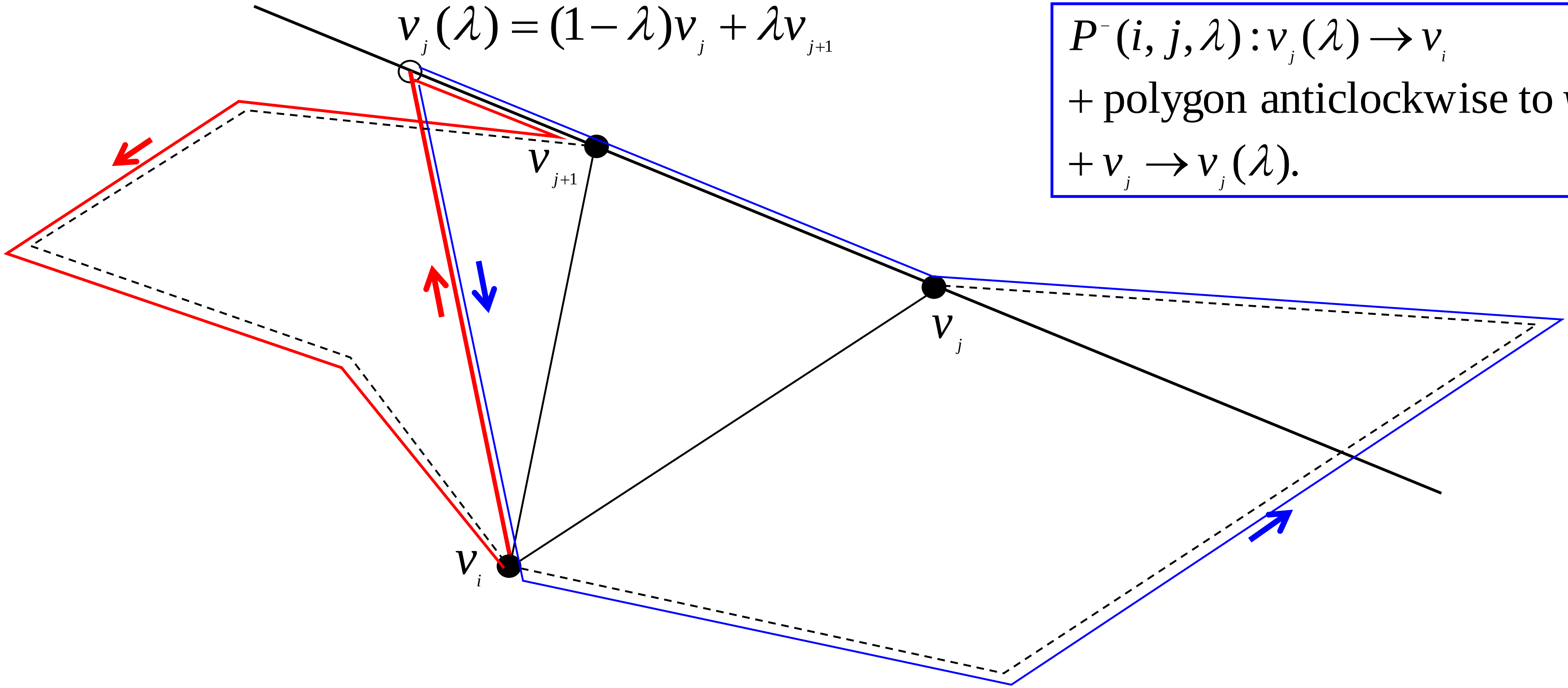


$r_{5,0} = 5/6$

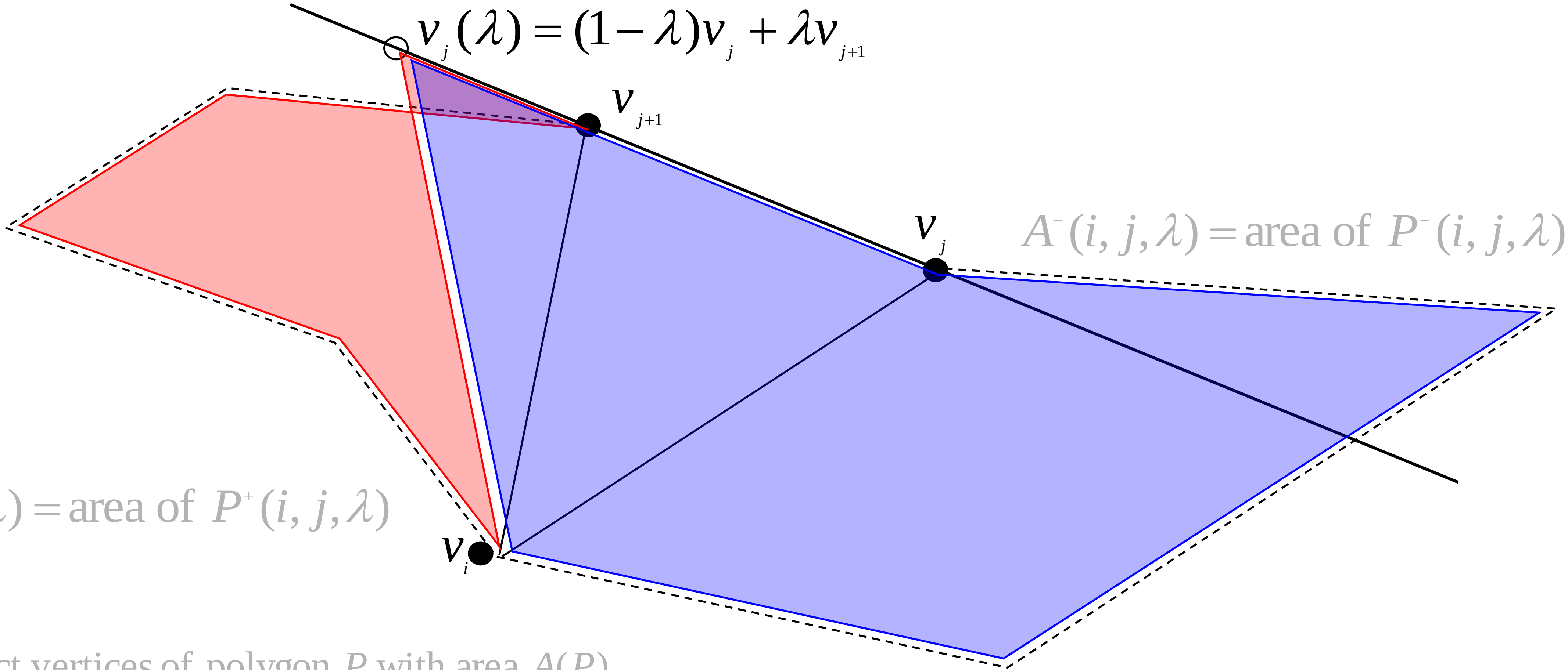
Bisection-convex pre-analysis IV
Auxiliary polygons

$P^{+}(i,j,\lambda):\ v_i \to v_j(\lambda) \to v_{j+1}$
$+$ polygon anticlockwise back to v_i .

$P^{-}(i,j,\lambda): v_j(\lambda) \to v_i$
$+$ polygon anticlockwise to v_j
$+ v_j \to v_j(\lambda)$.



Bisection-convex pre-analysis V
A theorem about auxiliary polygons



Let i, j be distinct vertices of polygon P with area $A(P)$

(1) for any real value of λ, $A^+(i,j,\lambda) + A^-(i,j,\lambda) = A(P)$

(2) if $r_{i,j} = \dfrac{A^+(i,j,0) - A^-(i,j,0)}{2\Delta_{i,j}}$ then $A^+(i,j,r_{i,j}) = A^-(i,j,r_{i,j}) = \dfrac{A(P)}{2}$

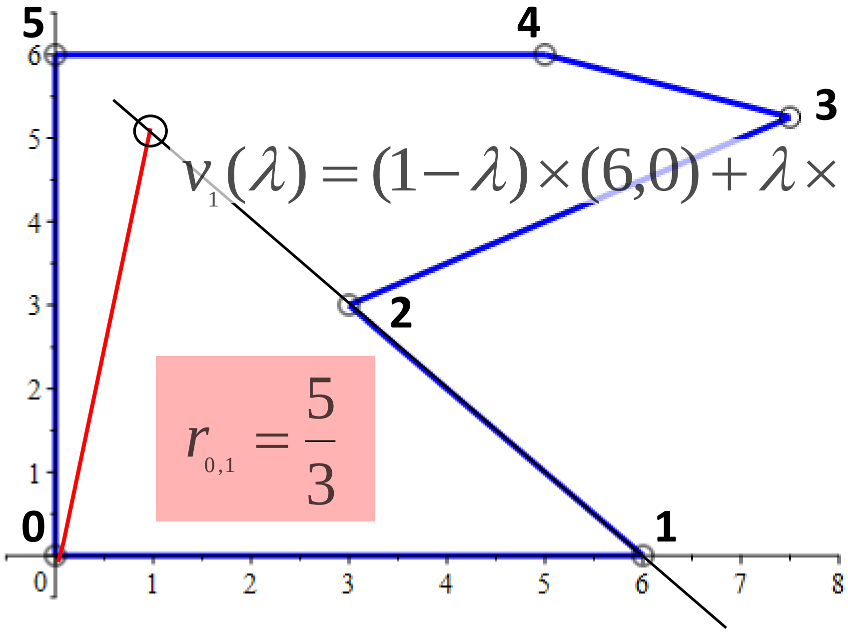
Bisection-convex pre-analysis VI
Theorem example

Let i, j be distinct vertices of polygon P with area $A(P)$

(1) for any real value of λ, $A^{+}(i,j,\lambda) + A^{-}(i,j,\lambda) = A(P)$

(2) if $r_{i,j} = \dfrac{A^{+}(i,j,0) - A^{-}(i,j,0)}{2\Delta_{i,j}}$ then

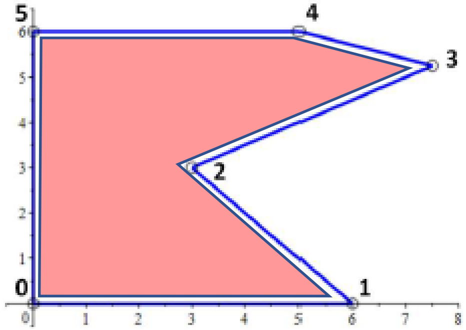
$$A^{+}(i,j,r_{i,j}) = A^{-}(i,j,r_{i,j}) = \frac{A(P)}{2}$$



$$v_1(\lambda) = (1-\lambda)\times(6,0) + \lambda\times(3,3)$$

$$r_{0,1} = \frac{5}{3}$$

$$R_P = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0 & \frac{5}{3} & -\frac{16}{9} & 1 & 0 & 0 \\ 1 & 0 & 0 & -\frac{40}{27} & \frac{67}{19} & \frac{6}{5} & \frac{1}{6} \\ 2 & -\frac{2}{3} & 0 & 0 & \frac{10}{3} & \frac{7}{5} & \frac{1}{3} \end{array}$$

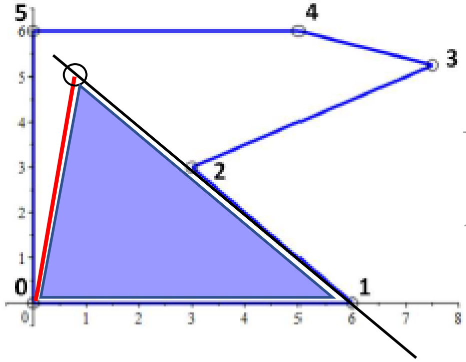


$A^{+}(0,1,0) = A(P) = 30$

$A^{-}(0,1,0) = 0$

$\Delta_{0,1} = 9$

$$r_{0,1} = \frac{A^{+}(0,1,0) - A^{-}(0,1,0)}{2\Delta_{i,j}} = \frac{30-0}{2\times 9} = \frac{5}{3}$$

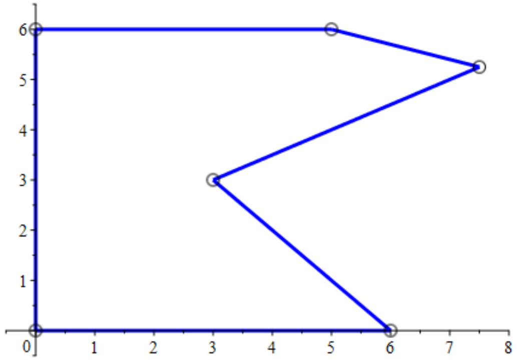


$v_1(5/3) = (1,5)$

$A^{-}(0,1,5/3) = 15 = A(P)/2$

Bisection-convex pre-analysis VII
Calculating $\boldsymbol{R_P}$

$$\Delta_P = \begin{array}{cc} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 9 & -\frac{27}{8} & \frac{75}{8} & 15 & 0 \\ 0 & 0 & -\frac{81}{8} & \frac{57}{8} & 15 & 18 \\ 9 & 0 & 0 & \frac{9}{2} & \frac{15}{2} & 9 \\ \frac{63}{4} & -\frac{81}{8} & 0 & 0 & \frac{15}{8} & \frac{45}{2} \\ 18 & -\frac{15}{2} & \frac{9}{2} & 0 & 0 & 15 \\ 18 & 0 & \frac{81}{8} & \frac{15}{8} & 0 & 0 \end{bmatrix} \end{array} \Rightarrow R_P = \begin{array}{cc} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & \frac{5}{3} & -\frac{16}{9} & 1 & 0 & 0 \\ 0 & 0 & -\frac{40}{27} & \frac{67}{19} & \frac{6}{5} & \frac{1}{6} \\ -\frac{2}{3} & 0 & 0 & \frac{10}{3} & \frac{7}{5} & \frac{1}{3} \\ -\frac{25}{42} & \frac{67}{27} & 0 & 0 & 8 & \frac{7}{12} \\ 0 & \frac{12}{5} & -\frac{7}{3} & 0 & 0 & 1 \\ \frac{5}{6} & 0 & -\frac{8}{27} & -7 & 0 & 0 \end{bmatrix} \end{array}$$



$$r_{i,j} = \begin{cases} \dfrac{A(P)}{2\Delta_{i,j}} & \text{if } j = i+1 \\ (r_{i,j} - 1)\dfrac{\Delta_{i,j-1}}{\Delta_{i,j}} & \text{if } j > i+1 \end{cases}$$

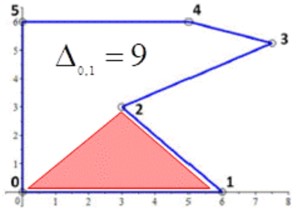
provided $\Delta_{i,j} \neq 0$

$$r_{0,1} = \frac{A(P)}{2\Delta_{0,1}} = \frac{30}{18} = \frac{5}{3}$$

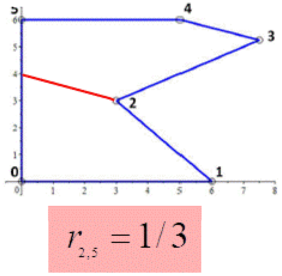
$$r_{0,2} = (r_{0,1} - 1)\frac{\Delta_{0,1}}{\Delta_{0,2}} = \left(\frac{5}{3} - 1\right) \times \frac{9}{-27/8} = -\frac{16}{9}$$

All-angles algorithm

1. Calculate triangle areas matrix (rank = 3, so linear time, in principle)

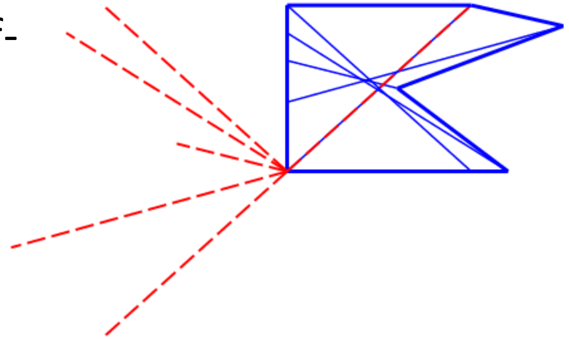


2. Calculate local bisections matrix (uses only triangle areas, so maybe linear time)



3. Test for bisection-convexity (can be done using only data from our two matrices)

4. If bisection-convex then find bisecting chords and arrange as a complete half-circle of direction vectors (otherwise resort to Shermer)



5. For each desired angle

   a. Locate angle on half-circle

   b. Calculate tilt of corresponding bisecting chord to get desired bisection