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Note

$2N$ Noncollinear Points Determine at Least $2N$ Directions

PETER UNGAR

Department of Mathematics, New York University, New York City, New York 10003

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We find sharp bounds for the number of moves required to bring a permutation to the form $n(n-1),..., 1$ if a move consists of inverting some increasing substrings.

If we invert every maximal increasing substring in each move we need at most $n-1$ moves.

If n is even and we start with $1, 2,..., n$ and we do not invert the entire permutation at once, then we need at least n moves.

The lower bound implies that when $n \geqslant 4$ is even, n points which are not collinear determine at least n different directions, as do $n+1$. These bounds are sharp.

What is the least number of different directions determined by n noncollinear points? This question seems to have been first considered by Scott [4] who observed that $2N+1$ points may determine as few as $2N$ directions (e.g., if they are the vertices and center of a regular $2N$-gon or if they form a centrally symmetric configuration with $2N-1$ of the points lying on a line) if they are allowed to be coplanar. He also gave a lower bound. The best lower bound for n points, prior to ours, was $[\frac{1}{2}n]$, obtained by Burton and Purdy [1]. The bound given here is best possible for plane sets but the problem remains open for nonplanar sets.

We employ the purely combinatorial approach devised for the planar problem by Goodman and Pollack [3]. They formulated two conjectures about the number of moves required to unravel permutations which we prove here as Theorems 1 and 2. Theorem 2 implies the bound given in the title.

The theorems are about changing a permutation of $1, 2,..., n$ to $n(n-1),..., 1$ by a sequence of *moves* each of which consists of inverting a disjoint set of increasing substrings. In the geometric context the permutations are the orthogonal projections of the set of points on a line.

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The line is rotated and a move occurs when the normal of the line passes through a direction determined by two or more of the points. (Goodman and Pollack [2] found that not every sequence of moves can be generated in this way.)

THEOREM 1. *If each move consists of inverting every maximal increasing substring, then we reach* $n(n-1),\ldots, 1$ *in at most* $n-1$ *moves.*

THEOREM 2. *Let* n *be even. If we start with* $12 \cdots n$ *and we do not invert the entire permutation in one piece, then the number of moves required to reach* $n(n-1) \cdots 1$ *is at least* n.

Proof of Theorem 1. It will suffice to prove that for any k, the numbers $\leqslant k$ will be to the right of the numbers $k+1,\ldots, n$ after at most $n-1$ moves. We shall call the numbers of the first set *small*, the others will be called *large*.

Let W_0 be the $s-l$ pattern associated with the initial permutation and let W_i be the pattern of the permutation obtained after move i. Let V_0 be $s \cdots sl \cdots l$ (k s's and $n-k$ l's). Let V_i be obtained from V_{i-l} by changing each sl into ls.

The patterns of the V_i are very simple. In the middle, s's and l's alternate and there may be strings of identical letters at the ends. The rightmost s reaches *box n* (the right–end position) after move $n-k$ which is also when the rightmost l starts moving. After $k-1$ more moves, the rightmost l arrives in box $n-k$ so that V_{n-1} is $l \cdots ls \cdots s$.

We claim that the jth l (from the left) in W_i is no farther from the left end than the jth l in V_i. This is clear for $i=0$. Suppose it is true for $i-1$. We prove it for i. We prove the assertion first for $j=1$. In the W-sequence the leftmost l moves to the left by at least 1 in each move until it reaches box 1. In the V-sequence it moves to the left by exactly 1 position in each move until it reaches box 1. Thus the statement is true for $j=1$. Let then j be the smallest value, if any, for which it fails. The jth l in W_{i-1} did not move then in move j, whereas the jth l in V_{i-1} moved by one place. Since the jth l in W_{i-1} is not to the right of the jth l in V_{i-1} they must be in the same place in these two words. Since, however, the jth l in W_{i-1} does not move in move i, it must be immediately preceded by the j-1th l, while in V_{i-1} it is preceded by an s. This would imply the $j-1$th l in W_{i-1} is to the right of the $j-1$th l in V_{i-1}, contrary to assumption. It follows now that W_{n-1} is $l \cdots ls \cdots s$ and Theorem 1 is proved.

Proof of Theorem 2. The argument we shall use will be easier to visualize if we allow ourselves to talk about *barriers* which we think of as separating consecutive boxes. The barrier which really interests us is the one between box no. $\frac{1}{2}n$ and box no. $\frac{1}{2}n+1$. (Remember n is even.) We call it the *center* *barrier*. Changing $12 \cdots n$ to $n(n-1) \cdots 1$ requires that each number be carried across the center barrier in some move. Thus there must be a total of at least n crossings of the center barrier.

Let us call a move in which at least one number crosses the center barrier a *crossing* move. A string straddling the center barrier, not necessarily symmetrically, is reversed in such a move. If d is the distance from the center barrier to the nearer end barrier of this string, then exactly $2d$ numbers will cross the center barrier in this move. Let $d_1, d_2, \ldots, d_t$ be the d's corresponding to the crossing moves. We have

$$2d_1 + \cdots + 2d_t \geqslant n. \tag{1}$$

Before the ith crossing move we have an increasing string extending a distance $\geqslant d_i$ on either side of the center barrier. Just after the move we have a decreasing string extending a distance $\geqslant d_i$ on either side of the center barrier. To build up the former and to dismantle the latter requires a certain number of noncrossing moves, which we shall count on the basis of the following two facts:

(I) A decreasing string can get shortened by not more than 1 at each end in one move.

(II) An increasing string can get longer by at most 1 at each end in one move.

The reason for (I) is that a move consists of inverting increasing strings. Therefore only the end members of a decreasing string can be moved in any move. The reason for (II) is that a number that is moved will be part of a decreasing string after the move. The interior members of an increasing string must therefore have been in place already before the move.

By (I) the center barrier will still be inside a decreasing string $d_i - 1$ moves after the ith crossing move. By (II) it will take at least d_{i+1} additional moves to build up an increasing string extending a distance $\geqslant d_{i+1}$ on either side of the center barrier. Thus there are at least $d_i + d_{i+1} - 1$ noncrossing moves between the ith and the $i+1$ th crossing moves.

Thus far we have t crossing moves and between them, $(d_1 + d_2 - 1) + \cdots + (d_{t-1} + d_t - 1)$ noncrossing moves. By (1) the number of these moves is $\geqslant n + 1 - d_1 - d_t$. The proof will be complete if we can show that the number of moves before the first crossing move plus the number of moves after the last crossing move is at least $d_t + d_1 - 1$. This is most easily done by bringing in more of the original context from which the problem we are solving was drawn. In that context the permutations represent projections of n points onto a rotating line, the first and the last permutation representing positions of the line 180° apart. The choice of the first position is arbitrary; it is more natural to continue the rotation both ways which gives an ultimately periodic sequence of permutations. This is the simple reason why the inequality which holds for the number of noncrossing moves between two crossing moves will also be valid for the number of noncrossing moves before the first and after the last crossing moves.

In the following discussion one should think of a permutation as just a linear ordering of n different objects whose nature does not imply any particular order. This is not a good way to think about permutations when one is discussing permutation groups but the group property plays no role here.

An *allowable sequence* of permutations,..., P_i, P_{i+1},..., is a periodic sequence of permutations with an even period p such that

(1) The move from P_i to P_{i+1} consists of reversing one or more nonoverlapping substrings of P_i.

(2) For any pair of objects a, b the succesive moves which reverse the order of a and b are exactly $\frac{1}{2}p$ apart.

Item (2) implies that $P_{i+p/2}$ is obtained by reversing P_i.

Each pair of elements is interchanged exactly once in the sequence of permutations in Theorem 2. Let p be twice the number of moves and denote the permutations by $P_0,..., P_{p/2}$. We extend this to an allowable sequence by defining $P_{i+p/2}$ to be P_i reversed.

Property (2) implies that if in any permutation of an allowable sequence we number the objects 1, 2,..., n (in the order in which they occur), then the next $\frac{1}{2}p$ moves consist of reversing increasing strings. Indeed, each pair whose order is inverted in these moves is inverted for the first time since the numbering.

The definition of a crossing move and the corresponding number d do not depend on the numbering of the objects. The total number of noncrossing moves before the first and after the last crossing move in the original sequence of permutations is the same as the number of noncrossing moves between the tth and the $t+1$th crossing moves in the extended sequence. There must be at least two crossing moves per half period because we are not allowed to reverse an entire permutation in one piece. Thus there is some half-period of the allowable sequence which contains both the tth and the $t+1$st crossing moves. We can number the objects in the order in which they occur in the first permutation of this half-period. The argument we gave above then assures us that there are at least $d_t + d_{t+1} - 1 = d_t + d_1 - 1$ noncrossing moves between these two crossing moves. This completes the proof of Theorem 2.

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