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# A Combinatorial Perspective on Algebraic Geometry*

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## 1. INTRODUCTION

Steinitz and Whitney were the first to observe that the axioms for linear independence in projective geometry and linear algebra can be given an abstract formulation. The resulting structure, now called a combinatorial geometry (definition below) has been intensively studied in the last few years. It turns out that an unexpected number of algebraic and combinatorial structures are combinatorial geometries. This observation has led to fruitful applications of abstract linear independence.

The purpose of the present work is to study algebraic varieties and certain algebraic subsets by the methods of combinatorial geometry. There arise in this study certain geometric configurations which resemble the classical geometries in many respects, and in which the axioms of abstract linear dependence still apply. Yet these axioms describe a phenomenon *completely different* from that to which we have become accustomed in affine and projective spaces.

This research follows the tradition of coordinate-free synthetic (therefore combinatorial) geometry, in the spirit of Steiner, Cremona, and Reye.

In order that the following discussion will be understandable for

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the algebraic geometer, we need to define here a combinatorial geometry. For a more detailed account, we refer to the treatise [2]. A *combinatorial geometry* $G$ is a set $X$ together with a closure operator satisfying the following properties:

(i) exchange: for any subset $A \subseteq X$ and elements $p, q \in X$, $q \notin \bar{A}$, $q \in \overline{A \cup p} \Rightarrow p \in \overline{A \cup q}$,

(ii) finite-basis: for any subset $A \subseteq X$, there is a finite subset $A_f \subseteq A$ with the same closure.

The closed subsets of $X$ are called *flats*. They form a *geometric lattice*, i.e., a complete lattice $L$ with no infinite chains and satisfying the following properties:

(i) atomistic: every flat $x \in L$ is the supremum of the set of atoms of $L$ beneath $x$,

(ii) semimodular: for any flats $x, y \in L$, if $x$ covers $x \wedge y$, then $x \vee y$ covers $y$.

It is an important fact that a geometric lattice is always represented as the lattice of flats of a combinatorial geometry. If there is no confusion, the associated geometric lattice of a combinatorial geometry $G$ will also be denoted by $G$, and the term *geometry* will simply be an abbreviation for either a combinatorial geometry or a geometric lattice. Each flat $x$ of a geometry $G$ has a well-defined *rank* $r(x)$, namely, the length of any maximal chain from the least element 0 of $G$ to $x$, satisfying the semimodular inequality:

$$r(x \wedge y) + r(x \vee y) \leqslant r(x) + r(y).$$

Flats of rank $k$ are called *k-flats*; those of rank 1, 2 and 3 are specifically called *points*, *lines* and *planes*. The *rank* of a geometry is defined to be the rank of its greatest flat. In a geometry of rank $N$, flats of rank $N - 1$ are called *co-points*.

A few classical geometries, specifically the projective, affine and Mobius geometries, are widely recognized as examples of combinatorial geometries. Some efforts have previously been made to expand this list of connections with classical geometry. For instance, Helgason [4] developed a combinatorial description of manifolds, and Wille [10] gave a lattice-theoretic characterization of incidence geometries generalizing the projective, affine and Mobius geometries.

In this paper we uncover a much broader spectrum of connections between combinatorial geometry and classical algebraic geometry: the theory of algebraic curves and surfaces.

Algebraic varieties can be realized as combinatorial objects in a natural way. For any field $F$ and any positive integer $n$, let $A^n$ be the set of points of an affine $n$-space over $F$. For any positive integer $k$, the zero-sets of polynomials of degree $k$ in $n$ variables, together with their intersections, form a combinatorial geometry on $A^n$. This geometry, denoted $A_k^n$, we call the *geometry of k-varieties in affine n-space over the field F*. Completing $A^n$ to the projective $n$-space $P^n$, and using homogeneous polynomials of degree $k$ in $n + 1$ variables, we obtain the *geometry* $P_k^n$ *of k-varieties in projective n-space over the field F*. (What we now say about varieties in affine space applies equally to varieties in the projective completion.)

The familiar notion of a linear transformation is generalized to a strong map in combinatorial geometry. A mapping $\sigma: G \to H$ between geometric lattices is a strong map if $\sigma$ is supremum-preserving and $\downarrow$-preserving. ($\downarrow$ denotes the relation "covering or equal to.") There is then an adjoint map $\sigma^{\Delta}: H \to G$ which is infimum-preserving, the two maps making a Galois connection $(\sigma, \sigma^{\Delta})$ [2]. Given a strong map $\sigma: G \to H$, the set of flats of $G$ which are maximum inverse images of flats of $H$ is a geometric lattice in the induced order of $G$, called a *quotient* of $G$. A *subgeometry* of a geometric lattice $G$ is a subset $K \subseteq G$ which is the image of some strong map into $G$. $K$ is a geometric lattice in the induced order of $G$; it is determined by the points of $G$ which it contains, and it consists precisely of those flats of $G$ which are suprema of points in $K$.

For each affine space $A^n$ over a field $F$, we have a sequence

$$A_1^n \leftarrow A_2^n \leftarrow A_3^n \leftarrow \cdots$$

of combinatorial geometries on the same set $A^n$ of points. Each geometry $A_i^n$ $(i > 1)$ is mapped onto its predecessor by a strong map. The limit of the sequence is the lattice of algebraic sets, a distributive lattice in which all finite sets of points are flats.

## 2. Vector Geometries

Before we turn to a study of those geometric lattices which arise in algebraic geometry, we record a few general theorems concerning vector geometries. A *vector geometry* is any subgeometry of a projective geometry $W$ of finite dimension over a field $F$. We will take some care to explain the connection between vector geometries and function space geometries, geometries defined by a vector space of functions from a set into a field. Function space geometries figure prominently in the literature of combinatorial geometries ("those associated with matrices" in Whitney [9], "chain groups" in Tutte [7], and "function space geometries" in [3]). Our effort here is to describe these geometries in terms of strong maps and, by so doing, to make explicit the notion of orthogonality, and the functorial relations which hold between categories of sets, vector spaces, and geometries.

The casual reader may prefer to read Theorem 2, then pass to Section 3 where we specialize to algebraic geometry.

A *function space geometry* $G(X, V)$ is a geometry on a set $X$ of points, whose flats are determined as follows by a vector space $V$ of functions from the set $X$ into a field $F$. A subset $A \subseteq X$ is a flat of the geometry $G(X, V)$ if and only if the set $A$ is an intersection

$$A = \bigcap_{\alpha} \operatorname{Ker} f_{\alpha}$$

of subsets $\operatorname{Ker} f_{\alpha} = f_{\alpha}^{-1}(0)$, for any family $\{f_{\alpha}\}$ of functions in $V$. The theory of function space geometries is developed in [2]. We give here a novel treatment of this theory in terms of strong maps, and outline a categorical approach to the subject.

THEOREM 1. *For any set $X$ and any finite-dimensional subspace $V$ of the vector space $F^X$ of functions from the set $X$ to a field $F$, the mapping $\sigma\colon \mathscr{B}(X) \to \mathscr{S}(V)$ from the Boolean algebra of subsets of $X$ to the lattice of subspaces of $V$ defined, for all subsets $A \subseteq X$, by*

$$\sigma(A) = \{f \in V; f(x) = 0 \ (\forall x \in A)\}$$

*takes suprema in $\mathscr{B}(X)$ to infima in $\mathscr{S}(V)$, and inverts the relation $\downarrow$, "covers or is equal to".*

*Proof.* The mapping $\sigma$ is well-defined, because $\sigma(A)$ is a linear subspace of $V$. For any family $\{A_{\alpha}\}$ of subsets of $X$, with $A = \bigcup_{\alpha} A_{\alpha}$,

$$\begin{aligned}
\sigma(A) &= \Big\{f \in V; f(x) = 0 \ \Big(\forall x \in \bigcup_{\alpha} A_{\alpha}\Big)\Big\} \\
&= \bigcap_{\alpha} \{f \in V; f(x) = 0 \ (\forall x \in A_{\alpha})\} \\
&= \bigcap_{\alpha} \sigma(A_{\alpha})
\end{aligned}$$

Assume $B$ covers $A$ in $\mathscr{B}(X)$, so $B = A \cup p$ for some element $p \notin A$. Say $\sigma(B) \subset Q \subseteq \sigma(A)$ for some *subspace* $Q$ of $V$. Then there is some function $f \in Q$ with $f(p) \neq 0$, and $f(x) = 0$ for all $x \in A$. Let $h$ be any function in $\sigma(A)$. Define a function $g$ by

$$g(x) = f(p)\, h(x) - h(p) f(x).$$

Then $g \in \sigma(B)$ because $h$ and $f$ are in $\sigma(A)$ and $g(p) = 0$. Since $f(p) \neq 0$, $h$ is in the subspace spanned by $g$ and $f$, both of which are in $Q$. Thus $\sigma(A) = Q$, and $\sigma(A)$ covers or is equal to $\sigma(B)$ in $\mathscr{S}(V)$. This completes the proof. ▌

Given such a set $X$ and a finite-dimensional subspace $V \subseteq F^X$, the map $\mathscr{B}(X) \xrightarrow{\sigma} \mathscr{S}(V)$ has an adjoint $\mathscr{S}(V) \xrightarrow{\sigma^\Delta} \mathscr{B}(X)$ defined by

$$\sigma^\Delta(W) = \{x \in X; f(x) = 0\ (\forall f \in W)\}$$

and characterized by the relation

$$A \subseteq \sigma^\Delta(W) \Leftrightarrow W \subseteq \sigma(A),$$

for all subsets $A \in \mathscr{B}(X)$ and for all subspaces $W \in \mathscr{S}(V)$. The pair $(\sigma, \sigma^\Delta)$ of maps forms a Galois connection [1], so the composites

$$\mathscr{B}(X) \xrightarrow{\sigma} \mathscr{S}(V) \xrightarrow{\sigma^\Delta} \mathscr{B}(X) \qquad \text{and} \qquad \mathscr{S}(V) \xrightarrow{\sigma^\Delta} \mathscr{B}(X) \xrightarrow{\sigma} \mathscr{S}(V)$$

are closure operators. We call the closed subsets of $X$ *algebraic sets.* (The closed subspaces might be called "localizable subspaces," but we shall have no occasion to use the term.) By the theory of Galois connections, the lattice $G(X, V)$ of algebraic sets is anti-isomorphic to the lattice of localizable subspaces.

THEOREM 2. *For any set $X$ and any $N$-dimensional subspace $V \subseteq F^X$, the lattice $G(X, V)$ of algebraic sets is geometric of rank $N$, and is representable as a subgeometry of the projective geometry of rank $N$ over $F$. The rank $r(A)$ of a subset $A \subseteq X$ is given by the formula*

$$r(X) - r(A) = \lambda(\sigma(A))$$

*where $\lambda(\sigma(A))$ is the rank of the subspace $\sigma(A) \subseteq V$ of functions in $V$ equal to zero on $A$.*

*Proof.* If we compose the map $\sigma$ with the lattice inversion $\mathscr{S}(V) \to$

$\mathscr{S}^{\text{opp}}(V)$, the composite $\sigma^{\text{opp}}$ is supremum-preserving and $(\downarrow)$-preserving. Further, it has the finite-basis property:

$$\text{for all subsets } A \subseteq X, \text{ there is a finite subset}$$
$$A_f \subseteq A \text{ such that } \sigma^{\text{opp}}(A_f) = \sigma^{\text{opp}}(A),$$

because chains of algebraic sets in $\mathscr{B}(X)$ are bounded in length by $N$. For any maximal chain

$$\varnothing = A_0 \subset A_1 \subset \cdots \subset A_n = \bar{A}$$

of algebraic sets, if $p_i$ is any point in the difference set $A_i \backslash A_{i-1}$, $(i = 1,\ldots, n)$, it follows that $A_i = \overline{A_{i-1} \cup p_i}$, so $\bar{A} = \overline{p_1 \cup \cdots \cup p_n}$, and $\sigma^{\text{opp}}$ has the finite-basis property.

Thus $\sigma^{\text{opp}}$ is a strong map from $\mathscr{B}(X)$ to $\mathscr{S}^{\text{opp}}(V)$. The lattice $G(X, V)$ is a geometric quotient of $\mathscr{B}(X)$, and is isomorphic to the image of $\sigma^{\text{opp}}$, a subgeometry of $\mathscr{S}^{\text{opp}}(V)$. The restriction of $\sigma^{\text{opp}}$ to the lattice $G(X, V)$ is rank-preserving, so

$$\begin{aligned} r(A) &= \lambda(\sigma^{\text{opp}}(A)) - \lambda(\sigma^{\text{opp}}(\varnothing)) \\ &= \lambda(V) - \lambda(\sigma(A)) = r(X) - \lambda(\sigma(A)). \quad \blacksquare \end{aligned}$$

In place of the strong map $\sigma^{\text{opp}}$ we might equally well have used the strong map $\sigma^*\colon \mathscr{B}(X) \twoheadrightarrow \mathscr{S}(V^*)$, which maps each subset $A \subseteq X$ to the subspace $\sigma^*(A)$ of linear functionals which are zero on all functions which are zero on $A$.

THEOREM 3. *Under the strong map $\sigma^*\colon \mathscr{B}(X) \twoheadrightarrow \mathscr{S}(V^*)$ the image $\sigma^*(A)$ of any subset $A \subseteq X$ is the subspace spanned by the evaluations $\pi_p$ for $p \in A$, where $\pi_p(f) = f(p)$. Thus the function space geometry $G(X, V)$ is isomorphic to the subgeometry of point-evaluations in $V^*$.*

*Proof.* The lattices $\mathscr{S}^{\text{opp}}(V)$ and $S(V^*)$ are isomorphic under the mapping $\alpha$, where, for any subspace $W \subseteq V$, $\alpha(W)$ is the set of linear functionals of $V$ which are zero on $W$. Thus $\sigma^* = \sigma\alpha$, and it suffices to prove that the image $\sigma^*(p)$ of a one-point subset $\{p\} \subseteq A$ is the subspace consisting of all multiples $k\pi_p$ of the evaluation $\pi_p$. If $g \in V^*$ is in $\sigma^*(p)$, then $g(f) = 0$ for all functions $f$ such that $f(p) = 0$. Assume $f_1(p) \neq 0$ for some $f_1 \in V$ (otherwise $g = 0 = 0\pi_p$). Let

$k = g(f_1)/f_1(p)$. For any function $f \in V$, $f_1(p)f - f(p)f_1$ is a linear function which is zero at $p$,

$$\begin{aligned} 0 &= g(f_1(p)f - f(p)f_1) \\ &= f_1(p)g(f) - f(p)g(f_1), \end{aligned}$$

and $\quad g(f) = kf(p) = k\pi_p(f)$. ▮

Theorem 2 proved that any function space geometry $G(X, V)$ is a vector geometry, a subgeometry of $\mathscr{S}^{\text{opp}}(V)$. We now prove the converse to Theorem 2. We use the fact that geometric dependence of points (rank one subspaces) in a projective geometry $\mathscr{S}(W)$ coincides with linear dependence of arbitrary non-zero vectors chosen from those rank one subspaces.

THEOREM 4. *Every vector geometry is a function space geometry.*

*Proof.* Assume a geometry $H$ on a set $X$ of points is a vector geometry, so that $X$ may be regarded as a set of non-zero vectors (no one a multiple of any other) in a finite dimensional vector space $W$. We assume, without loss of generality, that the set $X$ of vectors spans the vector space $W$.

For any subset $A \subseteq X$ and any point $q \in X$, $q \notin A$, the point $q$ is not in the $H$-closure of $A$ iff $q$ is not in the subspace of $W$ spanned by $A$, iff there is some linear functional $h \in W^*$ such that

$$h(p) = 0 \quad (\forall p \in A), \quad \text{and} \quad h(q) \neq 0.$$

Thus the geometry $H$ is the geometry $G(X, W^*)$. ▮

Between the category of sets and the category of vector spaces over a field $F$ we have the adjoint pair consisting of the forgetful functor "Under" and the functor "Free," which constructs the free vector space (formal linear combinations) on a set. For any vector space $W$, the evaluation map is a linear transformation from Free(Under($W$)) onto $W$, and has a kernel $K$ equal to the set of all expressions for 0 as a linear combination of vectors in $W$. Now, for any vector geometry $H \subseteq W$ on a set $X$ of points (vectors) on a finite-dimensional vector space $W$, the strong map

$$\mathscr{B}(X) \xrightarrow{\tau} \mathscr{S}(W)$$
$$\tau(A) = \text{subspace spanned by } A \text{ in } W,$$

factors as an injection $\phi$ of $\mathscr{B}(X)$ into the free vector space Free$(X, F)$ followed by the contraction $\kappa$ by the subspace $K$. That is:

$$\mathscr{B}(X) \xrightarrow{\phi} \mathscr{S}(\text{Free}(X, F)) \xrightarrow{\kappa} \mathscr{S}(W)$$

where for any subset $A \subseteq X$,

$$\phi(A) = \text{Free}(A, F) \subseteq \text{Free}(X, F)$$

and for any subspace $T$ of Free$(X, F)$,

$$\kappa(T) = K \vee T.$$

Similarly, for any function space geometry $G(X, V)$, the strong map

$$\mathscr{B}(X) \xrightarrow{\sigma} \mathscr{S}(V^*)$$

factors as

$$\mathscr{B}(X) \xrightarrow{\eta} \mathscr{S}((F^X)^*) \xrightarrow{\alpha} \mathscr{S}(V^*),$$

where

$$\eta(A) = \text{subspace of } (F^X)^* \text{ spanned by the evaluation maps } \pi_p\,, p \in A$$

and

$$\alpha(T) = J \vee T$$

where $J$ is the subspace of linear functionals $h$ on $F^X$ which are zero on all functions in $V$.

The function space geometry $G(X, V)$ is not the value of a bifunctor from a category of sets and a category of vector spaces, into the category of geometries. The reason is that the variables $X$ and $V$ are not independent: in a function space geometry, $V$ must be a finite dimensional subspace of $F^X$. A linear transformation of vector spaces need not preserve any of the structure of "zero sets" of coordinate representations. However, $G(X, V)$ is covariant in $X$ if the vector space $V$ is pulled back appropriately, and $G(X, V)$ is contravariant with respect to inclusion of finite-dimensional subspaces $V$ of $F^X$, if $X$ is fixed. This is the content of Theorems 5 and 6.

THEOREM 5. *For any function $X \xrightarrow{f} Y$ between two sets and for any* *finite-dimensional subspace $V$ of $F^Y$, let $\hat{f}(V)$ be the image of $V$ under the linear transformation $\hat{f}: F^Y \to F^X$, defined by*

$$\hat{f}(g) = fg \in F^X, \qquad \text{for all} \quad g \in F^Y.$$

*Then*

$$f: G(X, \hat{f}(V)) \to G(Y, V)$$

*is a strong map. The quotient geometry is isomorphic to the subgeometry of $G(Y, V)$ determined by the image of $f$.*

*Proof.* If a subset $B \subseteq Y$ is closed in $G(Y, V)$, let $A = f^{-1}(B)$, and let $p$ be a point in $X \backslash A$. $f(p) \notin B$, so there is a function $h \in V$ such that $h(B) = 0$, $h(f(p)) \neq 0$. Then the function $\hat{f}(h) \in \hat{f}(V)$ has value $h(f(A)) = 0$, $h(f(p)) \neq 0$. Thus $A = f^{-1}(B)$ is closed in $G(X, \hat{f}(V))$. $f$ is a strong map. ∎

THEOREM 6. *For any fixed set $X$, the geometry $G(X, V)$ is contravariant with respect to inclusion of finite-dimensional subspaces $V$ of $F^X$. In particular, if $W$ is a finite dimensional subspace of $F^X$ and if*

$$\mathscr{B}(X) \xrightarrow{\sigma} \mathscr{S}(V)$$

*is the map with quotient $G(X, V)$ as in Theorem 1, then the composite*

$$\mathscr{B}(X) \xrightarrow{\sigma} \mathscr{S}(V) \xrightarrow{\wedge W} \mathscr{S}(W)$$

*is the map with quotient $G(X, V \wedge W)$, a contraction of $G(X, V)$.*

*Proof.* If $T \subseteq V$ are finite-dimensional subspaces of $F$ the identity function on $X$ is a strong map from $G(X, V)$ to $G(X, T)$, because any $T$-variety is the set of common zeros of a set $A \subseteq T \subseteq V$ of functions, and is therefore also a $V$-variety. ∎

For each geometry $G$ on a *finite* set $X$, there is an *orthogonal* geometry $G^*$, also on $X$, determined by the condition on all subsets $A \subseteq X$ and points $p \notin A$,

$$p \in \bar{B}^* \qquad \text{if and only if} \quad p \notin \bar{A},$$

where $B = x \backslash (A \cup p)$. The following was remarked by Whitney [9].

THEOREM 7. *If the set $X$ is finite, then for any subspace $V \subseteq F^X$,*

$$G^*(X, V) = G(X, V^\perp)$$

*where $V^{\perp}$ is the orthogonal complement of the subspace $V$ relative to the basis for $F$ consisting of characteristic functions of one-element subsets $\{p\} \subseteq X$.*

*Proof.* Assume $A \subseteq X$, $p \notin A$, and $B = X \backslash (A \cup p)$. The point $p$ is in the orthogonal closure $\bar{B}^*$ if and only if $p \notin \bar{A}$, if and only if $V$ contains a function $g$ equal to zero on $A$, with $g(p) \neq 0$. The point $p$ is in the closure $\bar{B}^{\perp}$ of $B$ with respect to the orthogonal complement $V^{\perp}$ if and only if there is *no* function $f \in V^{\perp}$ equal to zero on $B$, with $f(p) \neq 0$. We must prove these statements are equivalent.

If the subspace $V$ contains a function $g$ equal to zero on $A$, with $g(p) \neq 0$, and if a function $f$ is equal to zero on $B$, with $f(p) \neq 0$, the inner product $(g,f)$ is equal to the product $g(p)f(p)$ which is non-zero in $F$. Thus $f \notin V^{\perp}$, and $p \in \bar{B}^*$ implies $p \in \bar{B}^{\perp}$.

Now assume $p \notin B^*$, $p \in \bar{B}^{\perp}$, and we shall arrive at a contradiction. For any subset $C \subseteq X$ let $\sigma(C)$ be the subspace of $F^X$ consisting of functions $f$ equal to zero on $C$. Then

$$\begin{array}{lll} p \notin \bar{B}^* & \text{implies} & V \wedge \sigma(A) \subseteq \sigma(A \cup p), \qquad \text{and} \\ p \in \bar{B}^{\perp} & \text{implies} & V^{\perp} \wedge \sigma(B) \subseteq \sigma(B \cup p). \end{array}$$

For any subset $C \subseteq X$, $(\sigma(C))^{\perp} = \sigma(X \backslash C)$, and for any subspaces $S$, $T$, $(S \cap T)^{\perp} = S^{\perp} \vee T^{\perp}$. Restating the conditions $p \notin \bar{B}^*$ and $p \in \bar{B}^{\perp}$ we have

$$V \wedge \sigma(A) \subseteq \sigma(A \cup p)$$
$$V \vee \sigma(A \cup p) \supseteq \sigma(A).$$

This is impossible, because $\sigma(A \cup p) \subsetneqq \sigma(A)$, and the lattice of subspaces is modular. ∎

If a geometry $G$ is a vector geometry, we know by Theorems 4, 7, then 2, that $G$ is a function space geometry, so $G^*$ is a function space geometry, and thus $G^*$ is a vector geometry. It is more convenient, however, if we are able to pass from a vector representation of $G$ directly to a vector representation of $G^*$. The following theorem establishes a method for doing so.

THEOREM 8. *If a geometry $G$ on $n$ points, rank $r$, has a vector representation over a field $F$, then it has a representation over $F$ in which a selected basis $B$ is represented by standard basis vectors, and the remaining points are represented by the row vectors of some $(n - r) \times r$ matrix.*

*The orthogonal geometry $G^*$ has a representation in which the points in $B$ are represented by the rows of the transpose $M^{\mathrm{tr}}$, the points of the complementary set $X \backslash B$ by standard basis vectors.*

*Proof.* Let $G$ be a vector geometry, with coordinatization

$$p_i \to (x_{i1}, \ldots, x_{ir}) \qquad i = 1, 2, 3, \ldots, n$$

where $r = \operatorname{rank} G$. Assume without loss of generality that $p_1, \ldots, p_r$ is a basis for $G$. Right-multiplying by the non-singular matrix

$$\begin{pmatrix} x_{11} & \cdots & x_{1r} \\ \vdots & & \vdots \\ x_{r1} & \cdots & x_{rr} \end{pmatrix}^{-1}$$

we obtain a coordinatization

$$p_i \to (y_{i1}, \ldots, y_{ir}) \qquad i = 1, 2, 3, \ldots, n$$

in which the basis points $p_1, \ldots, p_r$ are represented by

$$p_1 \to (1, 0, 0, \ldots), \quad \text{etc.,}$$

standard basis vectors. Let $M$ be the $(n - r) \times r$ matrix

$$\begin{pmatrix} y_{r+1,1} & \cdots & y_{r+1,r} \\ \vdots & & \vdots \\ y_{n,1} & & y_{n,r} \end{pmatrix}$$

Let $B$ be the basis $p_1, \ldots, p_r$, and let $A$ be any $r$-element set of points of $G$. Say the set $A_1 = A \cap B$ contains $k$ points, so the complementary set $A_2 = A \backslash A_1$ has $r - k$ points. The set $A$ is dependent in $G$ if and only if the determinant of the corresponding $r \times r$ matrix is zero. This matrix contains $k$ rows which are standard basis vectors, so the determinant is equal to that of the $(r - k) \times (r - k)$ matrix $M_A$ obtained by deleting those columns $i$ of $M$ such that $p_i \in A_1$, and keeping only those rows $i$ of $M$ such that $p_i \in A_2$.

Let $G'$ be the geometry coordinatized by

$$p_i \to (y_{r+1,i}, \ldots, y_{n,i}) \qquad i = 1, \ldots, r$$

$$p_i \to (0, \ldots, 1, \ldots, 0) \qquad i = r + 1, \ldots, n.$$

where the 1 occurs in the $i - r$ spot. The matrix of this coordinatization consists of $M^{tr}$, followed by an $(n-r) \times (n-r)$ identity matrix. The complement $X \backslash A$ of an $r$-element set $A$ is dependent in $G'$ if and only if the determinant of the corresponding $(n-r) \times (n-r)$ matrix is zero. This matrix contains $n-r+k$ rows which are standard basis vectors, so the determinant is equal to that of the $(r-k) \times (r-k)$ matrix obtained by deleting the columns $i$ of $M^{tr}$ such that $p_i \in X \backslash A_2$, keeping those rows $i$ of $M^{tr}$ such that $p_i \in X \backslash A_1$. The resulting $(r-k) \times (r-k)$ matrix is the transpose of the matrix $M_A$ obtained above.

Thus, for any $r$-element subset $A \subseteq X$, $A$ is dependent in $G$ if and only if the complementary set $X \backslash A$ is dependent in $G'$. Thus the complements of bases of $G$ are precisely the bases of $G'$, and $G'$ is isomorphic to the orthogonal geometry $G^*$. ▮

## 3. Algebraic Geometries

The approach to algebraic geometry by way of combinatorial geometries requires some restriction, preferably a natural restriction, upon the degree of polynomials used to define varieties. Relative to this degree restriction, definable varieties and definable algebraic sets acquire a well-defined geometric rank. It is this geometric rank which provides the answer to questions in enumerative geometry, for instance, "How many cubic curves pass through these seven points in the plane?" We pay a price for this information. For example, an algebraic set is defined, in the appropriate Galois connection, not by a radical ideal but by a degree-restrained subspace of such an ideal.

For any field $F$ and positive integers $n$ and $k$, let $F_{(k)}[x_1, ..., x_n]$ denote the vector space of polynomial functions in $n$ variables over $F$, with degree $\leqslant k$. We define a geometry $A_k^n$ (specifically $A_k^n(F)$), *the geometry of k-varieties of affine n-space* $A^n$, to be the function space geometry,

$$A_k^n = G(A^n, F_{(k)}[x_1, ..., x_n]).$$

The geometry $P_k^n$ of projective varieties is obtained as usual by projective completion (adding a copoint "at infinity") or by using homogeneous polynomials in $n+1$ variables.

Each set $Y$ of points in $n$-dimensional space $A^n$ has a well-defined rank in each of the geometries $A_k^n$. For instance, an affine line has rank 2 in the affine geometry $A_1^n$, rank $k+1$ in the geometry $A_k^n$ of $k$-varieties.

THEOREM 9. *The geometry $A_k{}^n$ of k-varieties in real n-space over the field $\mathbb{R}$ has rank $N$ equal to the binomial coefficient $\binom{n+k}{k}$. Each subset $Y \subseteq A^n$ has rank $r_k(Y)$ equal to $N$ minus the dimension of the subspace consisting of those polynomials of degree $\leqslant k$ which are zero at all points of $Y$.*

*Proof.* The real polynomial functions $f \in \mathbb{R}_{(k)}[x_1, \ldots, x_n]$ are determined independently by the coefficients of their distinct monomial terms. Binomial coefficients satisfy the identity

$$\binom{a}{b} = \sum_{j=0}^{b} \binom{a-j-1}{b-j}.$$

If we set $a = n + k$, $b = k$, we have

$$\binom{n+k}{k} = \sum_{i=0}^{k} \binom{n+k-i-1}{k-i}.$$

The ranks $r(A_k{}^n)$ also satisfy this recursion,

$$r(A_k{}^n) = \sum_{i=0}^{k} r(A_{k-i}^{n-1})$$

because each polynomial of degree $\leqslant k$ in $n$ variables may be factored with respect to the last variable, say $z$. The coefficient of $z^i$ in this expression is an arbitrary polynomial of degree $\leqslant k - i$ in $n - 1$ variables. The binomial coefficient and the rank agree (value $n + 1$) for all $n$ and $k = 1$, so they are equal.

The equality $r(y) = N - \lambda(\sigma(Y))$ is proven in Theorem 2. ∎

THEOREM 10. *The geometries $A_k{}^n$, for fixed n, are related by a sequence of surjective strong maps*

$$A_1{}^n \longleftarrow A_2{}^n \leftarrow A_3{}^n \longleftarrow \cdots$$

*As a consequence, the rank $r_k(Y)$ of any set $Y$ of points, $Y \subseteq A^n$, is monotonically increasing with respect to k. If $A^n \xrightarrow{e} A^m$ is an affine embedding of a space $A^n$ as an arbitrary affine flat in some higher dimensional space $A^m$, then the ranks of subsets of $A^n$ are unaffected:*

$$r_k(Y) = r_k(e(Y))$$

*for any set $Y$ of points in $A^n$.*

*Proof.* The maps $A_k{}^n \leftarrow A_{k+1}^n$ are induced by restricting the space of polynomials of degree $\leqslant k+1$ to the subspace of polynomials of degree $\leqslant k$, as in Theorem 6. From an affine $n$-flat $Y$ in affine $m$-space, $A^m$, local coordinates for $Y$ can be extended to a coordinatization of the entire space. Any polynomial in the first $n$ variables is also a polynomial in all $m$ variables (in which certain coefficients are zero.) Conversely, given a polynomial in $m$ variables, a polynomial with the same zeros on the flat $Y$ is obtained by setting the last $m - n$ variables equal to zero. Thus the $k$-varieties on the flat $Y$ are all intersections with $Y$ of $k$-varieties in $A^m$. Thus the flat $Y$, as a subgeometry of $A_k{}^m$, is isomorphic to $A_k{}^n$. ∎

In the following examples, the terms "conic," "quadric," etc. mean a point set definable by a polynomial of degree 2 but not by a polynomial of lower degree. The terms exclude those varieties determined by a single polynomial of degree 2 having a multiple factor.

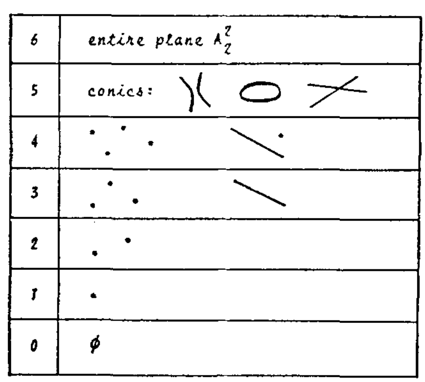


FIG. 1. Example 1.

EXAMPLE 1. $A_2{}^2(R)$, determined by conics in the real plane. This is a geometry of rank $\binom{2+2}{2} = 6$. We describe its flats as follows.

5-flats: These are the conics in the affine plane, namely: ellipses, parabolas, hyperbolas, and pairs of distinct lines.

4-flats: There are two kinds of 4-flats.

(a) 4 distinct points, no 3 of which are colinear.

(b) A line, together with a point not on it.

3-flats: There are two kinds of 3-flats.

(a) 3 noncolinear points.

(b) A line.

2-flats: These are the pairs of distinct points.

1-flats: These are the points in the affine plane.

The Mobius plane, a geometry of rank 4, is a quotient of this geometry $A_2^2$, obtained by restricting the space of polynomials

$$a_1x^2 + a_2y^2 + a_3xy + a_4x + a_5y + a_6,$$

to the rank 4 subspace defined by the conditions

$$a_1 = a_2, \qquad a_3 = 0.$$

If we replace the real field by the Galois field of two elements, then construct the geometry $A_2^2(GF(2))$, the space of polynomials is reduced in rank by the identities $x^2 = x$, $y^2 = y$, to a vector space of rank 4, so the geometry is the free geometry of four points, rank 4.

EXAMPLE 2. $A_3^2$, algebraic sets which are intersections of cubics in the real plane.

This is the geometry of 3-varieties in an affine plane. The rank of this geometry is $\binom{3+2}{2} = 10$.

9-flats: The copoints of this geometry are the cubic curves in the affine plane. Intersections of these curves yield all the flats of rank $\leqslant 8$.

8-flats: There are three kinds of 8-flats.

(a) A 9-point grid, that is nine distinct points which are the intersections of two cubic curves.

(b) 8 distinct points not contained in any 9-point grid, no 7 of which are on a conic, no 4 of which are colinear.

(c) A conic, together with 1 point not on it.

7-flats: There are three kinds of 7-flats.

(a) 7 points not contained in a conic, no 4 of which are colinear.

(b) A line, together with 3 distinct non-colinear points not on that line.

(c) A conic.

6-flats: (a) 6 points, no 4 of which are colinear.
(b) A line, together with 2 distinct points not on that line.

5-flats: (a) 5 points, no 4 of which are colinear.
(b) A line, together with a point not on it.

4-flats: (a) 4 points, not all colinear.
(b) A line.

3-flats: Any 3 points.

2-flats: Any 2 points.

1-flats: Any point in the affine plane.

We insert a remark of a general nature, before passing to the next example. Any subset of an affine space $A_0^n$, for example a spherical ball $B \subseteq A^n$, determines a subgeometry of each algebraic geometry $A_k^n$. Since there is an affine transformation of the space $A^n$ carrying the ball $B$ to any other ball $B'$, any two balls determine isomorphic subgeometries. As subgeometries of $A_1^n$, these are the classical hyperbolic geometries, such that through any point not on a line there is more than one parallel to that line.

EXAMPLE 3. $P_2^3$, varieties and algebraic sets determined by quadric surfaces in projective 3-space over the reals.
This geometry of 2-varieties in 3-space has rank 10.

9-flats: The copoints are the quadric surfaces, of which there are four types in the projective classification: oval surfaces (which are the projective completions of ellipsoids, elliptic paraboloids, and hyperboloids of two sheets of the affine 3-space), toroidal surfaces (which are the projective completions of hyperbolic paraboloids and hyperboloids of one sheet of the affine 3-space), cones (having a singular point) and pairs of distinct planes (meeting in a singular line).

8-flats: There are five kinds of 8-flats.
(a) 4 lines meeting in a point, no three of which are coplanar.
(b) A plane, together with a line not in that plane.
(c) The intersection of a quadric with two distinct planes, thus two plane conics *paired* in the sense that they lie on a single quadric.

(d) A space curve satisfying two irreducible quadratic equations. The curve has either one or two components, neither of which is contained in a plane.

7-flats: There are eight kinds of 7-flats:

(a) An 8-point array (for example the 8 vertices of a cube) on the intersection of three quadrics.

(b) 7 points, not contained in an 8-point array, no 5 of which are contained in a conic, and no 3 of which are colinear.

(c) A plane, together with a point not in that plane.

(d) A conic contained in a plane, together with 2 points skew to the conic, that is, points not in the plane, and such that the line joining them does not meet the conic.

(e) A conic contained in a plane, together with a line not in the plane, meeting the conic (at one point).

(f) A line, together with 4 points in general position, that is, they are not coplanar, no 3 are colinear, and no 2 are coplanar with the line.

(g) 3 lines meeting at a point, not coplanar.

(h) 3 lines, 2 of which are skew, both meeting the third line.

6-flats: There are five kinds of 6-flats.

(a) 6 points, no five of which are coplanar, no 3 of which are colinear.

(b) A plane.

(c) A conic, together with a point not on the plane of the conic.

(d) A line, together with 3 points in general position.

(e) 2 skew lines.

5-flats: There are three kinds of 5-flats.

(a) 5 points not coplanar, no 3 of which are colinear.

(b) A plane conic.

(c) A line, together with 2 points not coplanar with the line.

4-flats: (a) 4 points, no 3 colinear.

(b) A line, together with a point not on the line.

3-flats: (a) 3 noncolinear points.

(b) A line.

2-flats: Pairs of points.

1-flats: Points in the projective 3-space.

If we had chosen instead to describe the geometry $A_2{}^3$ of 3-varieties in the affine space (a spanning subgeometry of $P_2{}^3$), we would have had to take some care with respect to parallelism. For instance, a parabola $Q$ in a plane $S$, together with a line $M$ in a plane $T$, parallel to but not equal to $S$, constitutes a flat $Q \cup M$ of rank 7 if the line $M$ is parallel to the axis of the parabola, of rank 8 otherwise. The reason for this difference is that in the first instance, the line $M$ meets the parabola $Q$ at a point in the projective completion. Furthermore, if $p$ and $q$ are two points on the line $M$, then the subset $Q \cup p \cup q$ is a flat of rank 7 if $M$ is skew to the axis of $Q$, but has closure $Q \cup M$, also of rank 7, if $M$ is parallel to the axis.

The 8-point arrays of points in space occurring at rank 7 in $P_2{}^3$, as well as the 9-point planar arrays at rank 8 in $A_3{}^2$, are the first examples of an interesting class of configurations. In the geometry $A_3{}^3$ (rank 20) of varieties defined by cubic surfaces, we will find arrays 27 points at rank 17, obtained as the intersection of three cubic surfaces. There are also arrays of curves, such as the array of four lines at rank 8 in $A_2{}^3$, obtained as the intersection of two quadric surfaces, each of which consists of a pair of planes. In $A_3{}^3$ there is such an array of 9 lines, rank 18.

Semple and Roth study the representation of the quadrics of projective 3-space $P^3$ as points in projective 9-space $P_1{}^9$, determined by the vector space of polynomials of degree $\leqslant 2$, as in Theorem 2. The geometry $P_2{}^3$ is a subgeometry of the inverted modular geometric lattice $P_1{}^9$. The copoints of $P_2{}^3$, that is the quadric surfaces, are mapped to certain of the points of $P_1{}^9$: those which arise from polynomials with no repeated factors. These points generate a subgeometry of $P_1{}^9$, the flats of which are the linear varieties of quadric surfaces! The copoints of $P_2{}^3$ have become the points of a new geometry.

Subgeometries and quotients are among those structures derivable from any geometry. As Examples 4 and 5, we give two such derived structures.

EXAMPLE 4. A subgeometry of $A_1{}^3$.

Let $H$ be a hyperboloid of one sheet in affine space $A_1^3$. The points on the hyperboloid $H$ determine a subgeometry of $A_1^3$, a subgeometry which we denote also by $H$. Since the points of $H$ span $A_1^3$, this subgeometry also has rank 4. The flats of $H$ are:

3-flats: Plane intersections with $H$: these are ellipses, hyperbolas, or pairs of intersecting straight lines lying on the surface.

2-flats: (a) Pairs of points of $H$, no two on a line in $H$.

(b) Lines in $H$.

1-flats: Points of $H$.

Now the hyperboloid $H$, as a set of points, is also a flat of rank 9 in $A_2^3$, and each of the flats of $H$ as a subgeometry of $A_1^3$ is also a flat of $A_2^3$ in the interval $[0, H]$. The inclusion $H \subseteq A^3$ of point sets yields a strong map $[0, H] \to A_1^3$. This strong map is just the restriction to $[0, H]$ of the strong map

$$A_2^3 \twoheadrightarrow A_1^3$$

which we have used before, obtained by restricting the rank of polynomials available for defining varieties.

EXAMPLE 5. A quotient of $A_2^2$.

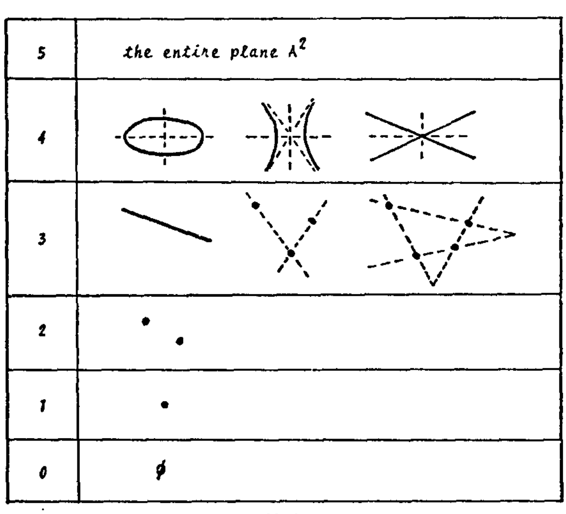


FIG. 2. Example 5.

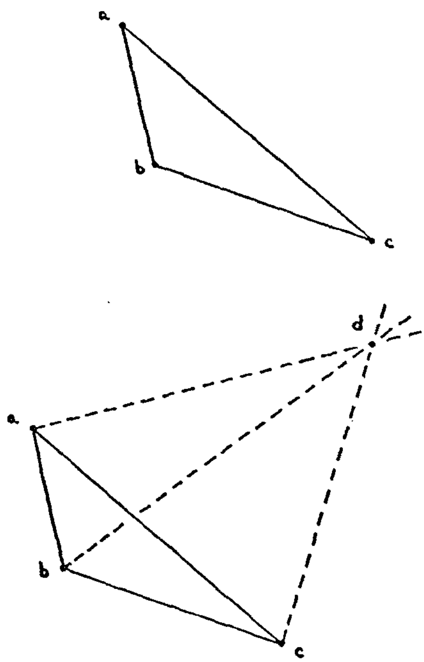


FIG. 3. Construction for Example 5.

Consider the vector space of polynomials having $a_2 = 0$ in the form

$$a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6 = 0.$$

This restriction $a_2 = 0$ determines a quotient of rank 5 in the geometry $A_2^2$. We determine the flats of the quotient geometry $R$, and the associated closure operator which maps each quadric variety to the smallest variety containing it and definable using only polynomials with $a_2 = 0$. The flats of the quotient are as follows:

4-flats: (a) Ellipses or hyperbolas with axes parallel to the $x$- and $y$-axes.

(b) Pairs of distinct lines with slopes negative to one another.

3-flats: (a) Lines.

(b) Triples of distinct points such that two sides of the triangle they form have slopes negative to one another. (We include the case of pairs of distinct vertical lines.)

(c) Sets of four distinct points such that opposite sides of the quadrangle they form have slopes negative to one another.

2-flats: Pairs of distinct points.

1-flats: Single points.

Note that the flats of rank 4 are those quadratic curves with vertical (and therefore horizontal) symmetry through their centers. At rank three we find a remarkable instance of the phenomenon of *associated points*. Certain three-point subsets are closed, but most possess a fourth point in their closure.

The construction of this fourth point is sketched in the accompanying figure. Assume we have three non-colinear points $a$, $b$, $c$. Let $x$ indicate any one of these points, and let $y$, $z$ indicate the other two. Construct three lines:

$$L_x \text{ passes through the point } x, \text{ with}$$
$$\text{slope } L_x = -\text{slope of line } y \vee z.$$

These three lines have a *common intersection* at the point $d$ which *depends upon* $a, b, c$ in this geometry. If it so happens that two of the three lines $a \vee b$, $b \vee c$, $c \vee a$ already have slopes negative to one another, the three-point set $a$, $b$, $c$ is closed.

EXAMPLE 6. *A Nonalgebraic construction.* The literature concerning combinatorial geometries contains any number of purely combinatorial nonalgebraic constructions. If these are applied to geometries whose flats are algebraic varieties, unusual geometries result. For instance, the Mobius geometry, rank 4, of circles and straight lines in the affine plane, is a quotient of $A_2^2$, rank 6. Higgs defined a lift construction which interpolates a geometry $L$ of rank 5 between $A_2^2$ and $M$:

$$A_2^2 \to L \to M.$$

The flats of this new geometry $L$ are indicated in the accompanying figure. It takes *four cocircular points* to span a circle in this geometry. Any four noncocircular points are closed. The geometry $L$ is not an algebraically-defined quotient of $A_2^2$, in the sense that it is not defined by any subspace of the vector space of quadratic polynomials. (If it were so defined, each copoint of $L$ would be the zero set of a single polynomial; but there are no polynomials equal to zero on precisely four distinct points.)

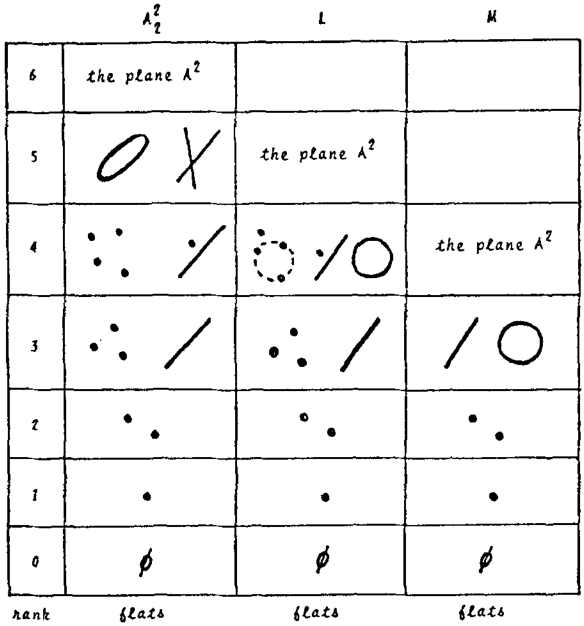


FIG. 4. Example 6.

EXAMPLE 7. *Grassmann line geometries.* So far we have looked at combinatorial geometries whose flats are algebraic varieties, but whose points were the points of an underlying projective space. As a final example, to indicate more accurately the great variety of combinatorial geometries which appear (if you look for them) in algebraic geometry, let us describe a combinatorial geometry whose *points* are the *lines* of a projective three-dimensional space. This is the geometry of linear varieties of lines in projective 3-space. Many books, among them Klein's "Vorlesungen uber hohere Geometrie," and treatises on algebraic geometry by Veblen and Young or by Semple and Roth, show how the lines of $P^3$, coordinatized as points in $P_1^5$, are those points on a quadric

$$x_1x_6 - x_2x_5 + x_3x_4 = 0,$$

the *Grassmannian,* and are thus a subgeometry of $P_1^5$, of rank 6. The points of this Grassmann geometry $G$ are the individual lines of $P^3$. A line in $G$ is either a pair of lines skew in $P^3$, or else a *flat pencil* of lines: those lying in a plane, and passing through some point on that plane. The planes (3-flats) of $G$ are of three types:

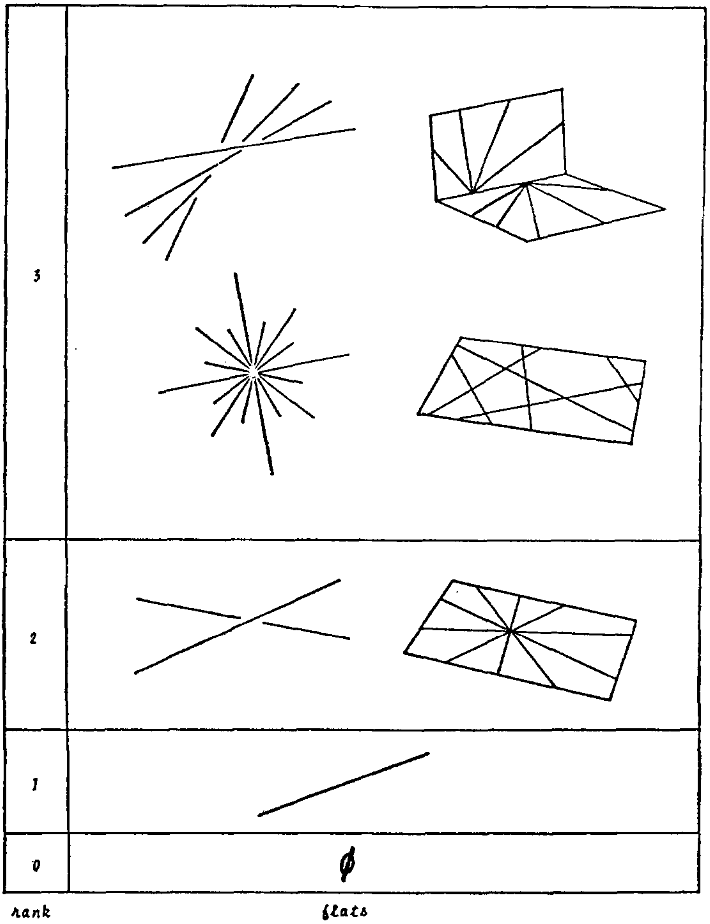


FIG. 5. Example 7. Line varieties of rules 0 to 3.

(3a) all lines in a plane,

(3b) all lines through a point,

(3c) the union of two flat pencils having a line in common, but lying in distinct planes and with distinct centers,

(3d) a regulus (one of the two 1-parameter families of mutually skew lines on a ruled quadric.)

Observe the significance of the main axiom of combinatorial geometries as we pass from rank 2 to rank 3 in the Grassmann geometry $G$ of lines.

If we add to a flat pencil (in a plane $P$, through a point $p$) a line $L$ not in that flat pencil, we generate a 3-flat of type

(3a) if $L$ lies in the plane $P$

(3b) if $L$ passes through the point $P$

(3c) otherwise.

Assume the line $L$ meets the plane $P$ in exactly a point $q \neq p$. Then $L$ together with line $p \vee q$ in the pencil generate a second pencil, and we obtain a flat of type 3c.

If we add a line $L$ to a 2-flat consisting of two skew lines $M$, $N$, we generate a 3-flat of type

(3c) if $L$ meets either $M$ or $N$,

(3d) otherwise.

We recommend the first case as an easy exercise. To generate a regulus, given three skew lines $L$, $M$, $N$, draw three distinct lines $D_i$ meeting $L$, $M$, $N$. (These is a unique such line through every point on $L$.) The regulus consists of those lines meeting all three lines $D_i$. (There is a unique such line through each point on $D_1$.)

EXAMPLE 7 (*cont'd*). *Line congruences.* Flats of rank 4 in the Grassmann geometry of lines are called *line congruences*. There are four types:

(4a) all lines in a plane, *or* through a point on that plane,

(4b) all lines which meet two fixed lines.

(4c) the union of a *linear* one-parameter family of flat pencils, all of which have a line in common,

(4d) a *linear spread*, that is a linear family of lines, containing exactly one line through every point in space.

The adjective *linear* is necessary to the description of line congruences of types 4c and 4d. It is clearly possible to nonlinearly parameterize the planes through a line $L$ with respect to the points on that line, and thereby to define a 1-parameter family of flat pencils which is not a line congruence. Similarly, the well-known construction, in which a single regulus in a spread is exchanged for its opposite regulus, produces a nonlinear spread. Line congruences are linear in the sense that they are *closed with respect to the operation of forming reguli* from three skew lines. We shall now take a moment to show how a basis for a line congruence determines the entire congruence.

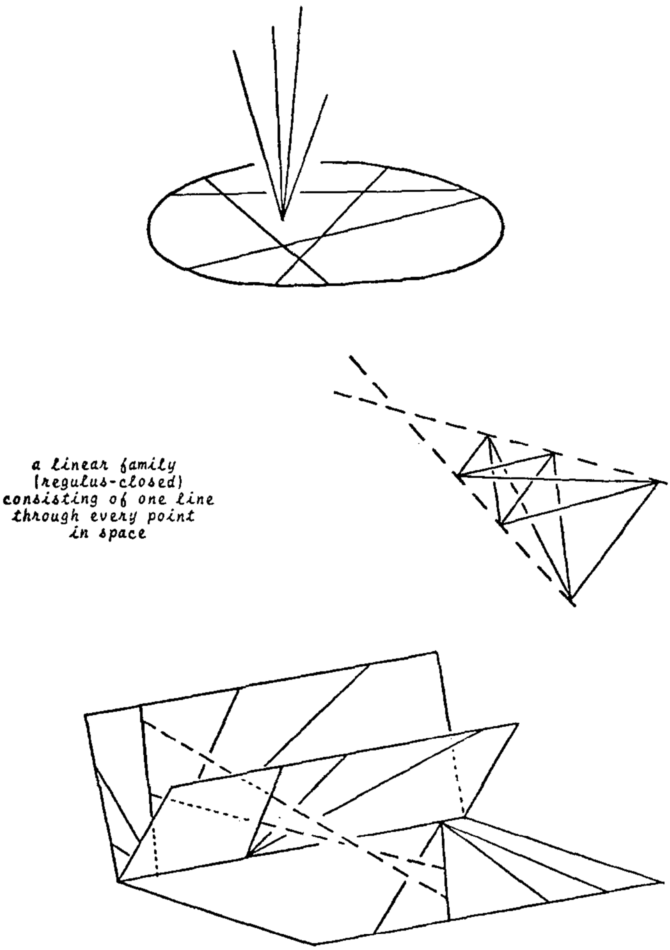


FIG. 6. Line varieties of rank 4: Line congruences.

Given a line congruence of type 4c, consisting of flat pencils with a line $L$ in common, let $p_i$ ($i = 1, 2, 3$) be distinct points on $L$, and let $T_i$ be lines of the congruence, distinct from $L$, through the points $p_i$. The four lines $L$, $T_1$, $T_2$, $T_3$ form a basis for the congruence. Construct two lines $D_1$, $D_2$ distinct from $L$, meeting the three lines $T_i$ (there is a unique such line through any point on $T_1\backslash L$). Now, given any point $q \notin L$, we construct the congruence line through $q$ as follows. Let $P$ be the plane $L \vee q$. The two points $D_i \wedge P$ generate a line meeting $L$ in a point $p$. The congruence line through $q$ is the line $p \vee q$. (In fact, the congruence contains the flat pencil through the point $p$ in the plane $P$.) Finally, given any point $p \in L$, we construct a flat pencil of congruence lines through $p$ in the plane $L \vee M$, where $M$ is the unique line through the point $p$ meeting both lines $D_1$ and $D_2$.

Now assume we have a congruence of type 4d, a linear spread. A basis for the spread will consist of *four distinct lines $L_i$ in the spread, such that no line in space meets all four lines $L_i$*. Given any point $p$ not on any of the lines $L_i$, we may construct the line $L$ through $p$ in the spread as follows. Using $L_1, L_2, L_3$, construct the regulus which contains them. This regulus will rule a surface $R$, say. If the point $p$ lies on the surface $R$, then the required line $L$ is the regulus line through $p$. Otherwise, $p \notin R$, and there is a cone $C$ of tangents to the surface $R$ through $p$. (The "inside" and the "outside" of this cone are distinguishable: lines through $p$ "outside" the cone are those meeting the surface $R$ in two points.) Since the line $L_4$ does not pass through the point $p$, there is some point $c \in L_4$ outside the cone $C$, and the line $p \vee c$ meets the surface $R$ in two distinct points $a$, $b$, through which there pass lines $L_a$, $L_b$ in the regulus on $R$. The line $p \vee c$ meets three lines of the spread, so the required line $L$ is the unique line through the point $p$ in the regulus generated by the lines $L_a$, $L_b$, and $L_4$.

EXAMPLE 7 (*cont'd*). *Line Complexes.* Maximal proper linear varieties of lines, those families of lines with rank 5 in the Grassmann geometry $G$, are called *line complexes*. These are two types of line complexes:

(5a) singular: all lines which meet one fixed line,

(5b) nonsingular: a family of lines containing, through every point in space, precisely a flat pencil of lines.

Note that we do not use the adjective "linear" in describing non-singular line complexes. The fact that *every such family of lines is linear* is an important consequence of Sylvester's theorem ([6, I, p. 323–4]).

It is in the detailed study of nonsingular line complexes that we encounter connections with physical theory, and with mechanics in particular. These connections have been lost to the current literature, through historical accident. But Felix Klein's book "Geometry" (Dover, 1939) refers us to Sir Robert Ball's monograph (Cambridge, 1900) entitled "Theory of Screws," and states that the lines of a line complex are precisely those lines orthogonal to some right-angle screw motion of space. (The remarkable fact about such screw motions is that if a line is orthogonal to the motion at *one* of its points, then it is *everywhere* orthogonal to the motion: an apparently local description is actually global.)

## Conclusion

In conclusion, we can only hope you share our pleasure in finding that a theory which we had thought applied only to linear dependence has in fact a broader scope, and includes objects, constructions, and invariants of classical algebraic geometry.

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