

A new upper bound on the clique number of a strongly regular graph

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Introduction

All graphs in this talk are finite and undirected, with no loops and no multiple edges.

Definition

A **clique** in a graph Γ is a set of pairwise adjacent vertices. The **clique number** of Γ is the size of a largest clique in that graph, and is denoted by $\omega(\Gamma)$.

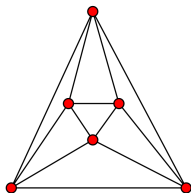
- I want to talk about how Gary Greaves (NTU, Singapore) and I used algebra to determine new upper bounds for $\omega(\Gamma)$, for strongly regular graphs Γ with given parameters.
- Our main tools were “clique adjacency polynomials” and Gröbner bases.

Edge-regular graphs

Definition

A graph Γ is **edge-regular** with **parameters** (v, k, λ) if Γ has exactly $v > 0$ vertices, is regular of valency $k > 0$, and every edge lies in exactly λ triangles.

For example, the octahedron graph $\Delta :=$



(copied from Wikimedia Commons) is edge-regular, with parameters $(6, 4, 2)$, and $\omega(\Delta) = 3$.

The clique adjacency polynomial

Definition

The **clique adjacency polynomial** of an edge-regular graph Γ with parameters (v, k, λ) is $C_{v,k,\lambda}(x, y) :=$

$$x(x+1)(v-y) - 2xy(k-y+1) + y(y-1)(\lambda-y+2).$$

This polynomial is a special case of the “block intersection polynomials” introduced by Peter Cameron and me in 2007, and further studied by me since then. Block intersection polynomials have found applications in design theory, graph theory, and the theory of tournaments.

The theory of block intersection polynomials can be used to prove the following:

Theorem (S., 2010)

Let Γ be an edge-regular graph with parameters (v, k, λ) , suppose Γ has a clique S of size $s \geq 2$, and let n_i be the number of vertices of Γ not in S adjacent to exactly i vertices of S ($i = 0, \dots, s$). Then:

- ① For $j = 0, 1, 2$,

$$\sum_{i=0}^s \binom{i}{j} n_i = \binom{s}{j} \lambda_j,$$

where $\lambda_0 = v - s$, $\lambda_1 = k - s + 1$, and $\lambda_2 = \lambda - s + 2$;

- ② $C_{v,k,\lambda}(x, s) = \sum_{i=0}^s (i - x)(i - x - 1)n_i$;
- ③ $C_{v,k,\lambda}(m, s) \geq 0$ for every integer m .

Bounding the clique number of an edge-regular graph

A clique adjacency polynomial can be used to determine an upper bound on the clique number of an edge-regular graph with given parameters, as follows.

Let Γ be an edge-regular graph with parameters (v, k, λ) . Define the **clique adjacency bound** $c_{v,k,\lambda}$ for Γ to be the least integer $c > 1$ such that $C_{v,k,\lambda}(m, c + 1) < 0$ for some integer m . Then $\omega(\Gamma) \leq c_{v,k,\lambda}$.

Such a clique adjacency bound always exists, and is easy to calculate.

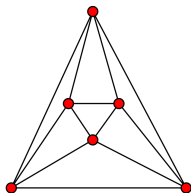
For example, $C_{6,4,2}(1, 4) = -4$, so $c_{6,4,2} = 3$.

Strongly regular graphs

Definition

A graph Γ is **strongly regular** with **parameters** (v, k, λ, μ) if Γ is a non-complete edge-regular graph with parameters (v, k, λ) , such that every pair of distinct non-adjacent vertices have exactly μ common neighbours.

For example, the octahedron graph



(copied from Wikimedia Commons) is strongly regular, with parameters $(6, 4, 2, 4)$.

Strongly regular graphs

Let Γ be a strongly regular graph with parameters (v, k, λ, μ) . It is well-known that Γ has at most three distinct eigenvalues $k > r \geq s$, and that

$$(v - k - 1)\mu = k(k - \lambda - 1), \quad \lambda - \mu = r + s, \quad \mu - k = rs.$$

In particular, the eigenvalues of Γ are determined by its parameters.

In his famous PhD thesis (1973), Ph. Delsarte proved that

$$\omega(\Gamma) \leq \lfloor 1 - k/s \rfloor.$$

We refer to this bound as the **Delsarte bound**. (It is sometimes called the Delsarte-Hoffman bound.)

For example, the least eigenvalue of the octahedron graph is -2 , and so the Delsarte bound is $3 = \lfloor 1 - 4/(-2) \rfloor$, the same as the clique adjacency bound.

Type I and type II strongly regular graphs

Following Cameron and Van Lint, if the parameters (v, k, λ, μ) of a strongly regular graph Γ satisfy

$$k = (v - 1)/2, \quad \lambda = (v - 5)/4, \quad \mu = (v - 1)/4,$$

then we say Γ is **type I**. (Type-I strongly regular graphs are also known as conference graphs.) The Delsarte bound for a v -vertex type I strongly regular graph Γ translates to $\omega(\Gamma) \leq \lfloor \sqrt{v} \rfloor$.

Strongly regular graphs all of whose eigenvalues are integers are called **type II**.

Every strongly regular graph is type I, type II, or both type I and type II.

The clique adjacency bound vs the Delsarte bound

Experimentation found that for a strongly regular graph with given parameters, the clique adjacency bound seemed always to be at least as good as the Delsarte bound, and was sometimes strictly better.

Joint work with Gary Greaves started at Anton Betten's CoCoA15 conference in summer 2015 to try to prove this.

Some of our results follow. We found it useful to treat type I and type II graphs separately.

Theorem (Greaves and S., 2016)

Let Γ be a type-I strongly regular graph with parameters (v, k, λ, μ) . Then

$$C_{v,k,\lambda}(\lfloor (\sqrt{v} - 1)/2 \rfloor, \lfloor \sqrt{v} + 1 \rfloor) < 0.$$

Theorem (Greaves and S., 2016)

Let Γ be a type-II strongly regular graph with parameters (v, k, λ, μ) and eigenvalues $k > r \geq s$. Then

$$C_{v,k,\lambda}(\lfloor -\mu/s \rfloor, \lfloor 2 - k/s \rfloor) < 0.$$

Hence, for all strongly regular graphs, the clique adjacency bound is at least as good as the Delsarte bound. How often is it better?

We next give sufficient conditions for the clique adjacency bound to be strictly better than the Delsarte bound.

Theorem (Greaves and S., 2016)

Let Γ be a type-I strongly regular graph with parameters (v, k, λ, μ) , and suppose that

$$0 < \sqrt{v}/2 - \lfloor \sqrt{v}/2 \rfloor < 1/4 + (\sqrt{v} - \sqrt{v + 5/4})/2.$$

Then $C_{v,k,\lambda}(\lfloor (\sqrt{v} - 1)/2 \rfloor, \lfloor \sqrt{v} \rfloor) < 0$, and so $\omega(\Gamma) \leq \lfloor \sqrt{v} \rfloor - 1$.

We can apply this theorem to obtain an improvement on the Delsarte bound for about a quarter of all “feasible” parameter tuples for type-I strongly regular graphs, including infinitely many parameter tuples of the form $(p, (p - 1)/2, (p - 5)/4, (p - 1)/4)$, with p a prime congruent to 1 mod 4, corresponding to Paley graphs on a prime number of vertices.

Theorem (Greaves and S., 2016)

Let Γ be a co-connected type-II strongly regular graph with parameters (v, k, λ, μ) and eigenvalues $k > r \geq s$. Suppose that

$$0 < -k/s - \lfloor -k/s \rfloor < 1 - (r^2 + r)/(v - 2k + \lambda).$$

Then $C_{v,k,\lambda}(\lfloor -\mu/s \rfloor, \lfloor 1 - k/s \rfloor) < 0$, and so $\omega(\Gamma) \leq \lfloor -k/s \rfloor$.

Experimental evidence suggests that we can apply this theorem to obtain an improvement on the Delsarte bound for about an eighth of all “feasible” parameter tuples for type-II strongly regular graphs.

Algebraic computations

Many of our proofs require the establishment of one or more identities involving the clique adjacency polynomial and the parameters and eigenvalues of a strongly regular graph.

For this, we usually used a Gröbner basis in a standard way to effectively compute in a quotient of a ring of multivariate polynomials over the rationals.

Our calculations were done using Maple (taking a total of 0.16 seconds of CPU time), and were checked using Magma.

A sample computation

We start up Maple (Version 18) and assign to C the clique adjacency polynomial.

```
> C:=x*(x+1)*(v-y)-2*x*y*(k-y+1)+y*(y-1)*(lambda-y+2):
```

We then make a set of relators, which evaluate to 0 on the parameters (v, k, λ, μ) and eigenvalues r, s (with $k > r \geq s$) of a strongly regular graph.

```
> srg_rels:=  
> {(v-k-1)*mu-k*(k-lambda-1), (lambda-mu)-(r+s), (mu-k)-r*s}:
```

Let $R = \mathbb{Q}[t, v, k, \lambda, \mu, r, s]$ be the ring of polynomials over \mathbb{Q} in the indeterminates $t, v, k, \lambda, \mu, r, s$, and let I be the ideal of R generated by `srg_rels`. We will calculate and employ Gröbner bases to work in the factor ring R/I .

We set the monomial ordering for the Gröbner basis calculations to be the Maple `tdeg` ordering, more commonly called the `grevlex` ordering, with the indeterminates ordered as $t > v > k > \lambda > \mu > r > s$.

```
> ordering:=tdeg(t,v,k,lambda,mu,r,s):
```

Then we compute a Gröbner basis G for I , with respect to this ordering.

```
> G:=Groebner[Basis](srg_rels,ordering):
```

For the record,

$$G = [\lambda - \mu - r - s, rs + k - \mu, k^2 - kr - ks - \mu v - k + \mu].$$

Now here is the verification of a useful identity:

$$C_{v,k,\lambda}(-\mu/s - t, 1 - k/s - t) = t((v - 2k + \lambda)(t - 1) + r(r + 1)),$$

for all strongly regular graphs with parameters (v, k, λ, μ) and eigenvalues $k > r \geq s$, and all real numbers t .

```
> Groebner [NormalForm] (  
>   expand(s^3*(eval(C, [x=-mu/s-t, y=1-k/s-t])  
>     - t*((v-2*k+lambda)*(t-1)+r*(r+1))))), G, ordering);  
0
```

A final comment

As far as I am aware, when there exists a strongly regular graph with parameters (v, k, λ, μ) , with $k < v/2$, then there is at least one strongly regular graph with these parameters having clique number equal to the clique adjacency bound.

This has been verified by my PhD student Rhys Evans for all strongly regular graphs having at most 40 vertices.

Some references

- G.R.W. Greaves and L.H. Soicher, On the clique number of a strongly regular graph, submitted for publication, 2016. Preprint available at <http://arxiv.org/abs/1604.08299>
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- L.H. Soicher, On cliques in edge-regular graphs, *J. Algebra* **421** (2015), 260–267.