

Graph homomorphisms II: some examples

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Abstract

In this talk we will discuss some examples of graph homomorphisms. More precisely, the graph parameters which can be represented by counting the graph homomorphisms. The main reference is Section 2 in [2].

1 Introduction

In this note, we will study some of explicit examples about graph homomorphism, which provides a useful language and motivation to continue our study about [2]. The basic setting is as follows: $G = (V(G), E(G))$ is a simple graph unless stated otherwise, $\phi : G \rightarrow H$ is a homomorphism from G to H and $hom(G, H)$ is the number of homomorphisms from G to H .

In fact, we should consider the homomorphisms both “from” G and “to” G . The basic scheme in the paper [2] is:

$$F \rightarrow G \rightarrow H.$$

Given a (large, simple) graph G , we can study its local structure by counting of various “small” graphs F into G ; and its global structure by counting its homomorphisms into various small graph H . Roughly speaking, we can get some information about G via “probing from the left with F ”, which is related to property testing. On the other hand, we can “probing from the right with H , which is related to statistical physics. Informally, H is also called the template (Model) of G . One useful observation is that any $\phi : G \rightarrow H$ gives a partition on $V(G)$ via the fibres of ϕ .

2 Weighted and unweighted

A *weighted graph* H is a graph with a positive real weight $\alpha_H(i)$ associated with each node i and a real weight $\beta_H(i, j)$ associated with each edge ij .

An edge with weight 0 will play the same as role as no edge between those nodes, so we could assume that we only consider weighted complete graphs with loops at all nodes. An unweighted graph is a weighted graph where all the nodeweights and edgeweights are 1.

Let G and H be two weighted graphs. To every map $\phi : V(G) \rightarrow V(H)$, we assign the weight:

$$hom_\phi(G, H) = \prod_{uv \in E(G)} [\beta_{\phi(u)\phi(v)}(H)]^{\beta_{uv}(G)} \quad (1)$$

here ($0^0 = 1$). We then define

$$hom(G, H) = \sum_{\phi: V(G) \rightarrow V(H)} \alpha_\phi hom_\phi(G, H) \quad (2)$$

where

$$\alpha_\phi = \prod_{u \in V(G)} [\alpha_{\phi(u)}(H)]^{\alpha_u(G)}. \quad (3)$$

We'll use this definition most often in the case when G is a simple unweighted graph, so that:

$$\alpha_\phi = \prod_{u \in V(G)} [\alpha_{\phi(u)}(H)].$$

and

$$hom_\phi(G, H) = \prod_{uv \in E(G)} [\beta_{\phi(u)\phi(v)}(H)]$$

3 Example

In this section we will present some examples about $hom(F, G)$ ($hom(G, H)$), where F (H) is a single graph or a subfamily of finite graphs. The adjacent matrix of G is denoted by $A(G) = (a_{ij})$, while $A^k = A(G^k) = ({}^k a_{ij})$ is the adjacent matrix of G^k , the k power graph of G . More precisely, ${}^k a_{ij}$ is the number of walks from i to j and of length k .

3.1 Left & unweighted

Example 1. Let F be a point, then $hom(F, G) = |V(G)|$.

Let F be K_2 , then $hom(F, G) = 2|E(G)|$.

Let F be K_3 , then $hom(F, G) = 6 \times \#$ of triangles. (note: here G is assume to be simple.)

Example 2. Let F be a k -path P_k , then $hom(P_k, G) = \#$ of walks with length $k - 1$. Here (v_i, v_j, v_k) and (v_k, v_j, v_i) should be treated as two different walks of length 3. In other words, $hom(P_k, G) = \sum_{i,j} \binom{k-1}{i,j} a_{ij}$.

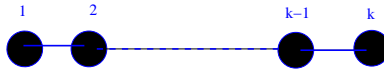


Figure 1: the path on k nodes

Example 3. Let S_k be the star on k nodes, then $hom(S_k, G) = \sum_{i=1}^n d_i^{k-1}$ where d_i is the degree of $i \in V(G)$. Hence $hom(S_k, G)^{1/(k-1)}$ tends to the maximum degree of G as $k \rightarrow \infty$.

Proof. Denote the vertices set of S_k by $\{1, 2, \dots, k\}$. Pick any $v \in V(G)$, and study the number of homomorphisms of $\phi : S_k \rightarrow G$ s.t $\phi(1) = v$. For each vertex of S_k other than 1, there are d_v different choices in $V(G)$ as its image where d_v is the degree of v . Therefore we have totally d_v^{k-1} such homomorphisms. On the other hand, the image of 1 can run through all vertices of G , therefore the number of homomorphisms between S_k and G is $\sum_{i=1}^n d_i^{k-1}$. □

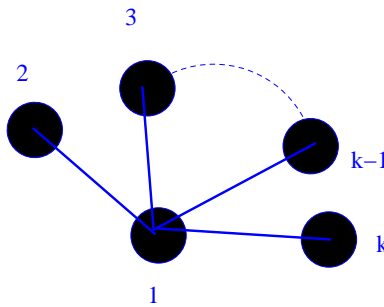


Figure 2: the star on k nodes

Example 4. Let C_k be the cycles on k nodes, then

$$\text{hom}(F, G) = \sum_{i=1}^n {}^k a_{ii} = \sum_{i=1}^n \lambda_i^k$$

where λ_i is the eigenvalue of G .

Proof. In this example, G is not necessarily to be simple. Firstly we claim the number of loops in G is equal to the sum of its eigenvalues. That is because this sum is the trace of A_G , the adjacent matrix of G , and we know the trace of A_G counts the number of loops in G . Secondly we note that $\sum_{i=1}^n \lambda_i^k$ is the trace of A_{G^k} where $V(G^k) = V(G)$ and $(i, j) \in E(G^k)$ iff there is exactly one path of length k from i to j in the graph G . More precisely, $A_{G^k} = \underbrace{A(G) \times A(G) \cdots \times A(G)}_k$. On the other hand, the number of loops in G^k is the same as $\text{hom}(C_k, G)$, which complete the proof. \square

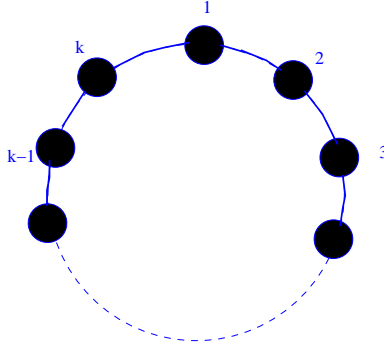


Figure 3: the cycles on k nodes

Example 5. (Random graphs) Let $G(n, p)$ be a random graph with n nodes and edgedensity p . Then for every simple graph F with k nodes,

$$E(\text{hom}(F, G)) = (1 + o(1))n^k p^{|E(F)|} \quad (n \rightarrow \infty).$$

Proof. Given any map ϕ from F to G , denote the probability that ϕ is a homomorphism by ρ . Then we know $\rho = p^{|E(F)|}$ when ϕ is 1-1 and $p^{|E(F)|} \leq \rho \leq 1$ otherwise. On the other hand, the number of the 1-1 map from F to G is $n(n-1) \cdots (n-k+1)$ while the total number of the maps is n^k . Put all these together, we have:

$$E(\text{hom}(F, G)) = n(n-1) \cdots (n-k+1)p^{|E(F)|} + (n^k - n(n-1) \cdots (n-k+1))t$$

where $p^{|E(F)|} \leq t \leq 1$. Let $\delta = t - p^{|E(F)|}$. Then $0 \leq \delta \leq 1$ and we have:

$$\begin{aligned} E(\text{hom}(F, G)) &= n^k p^{|E(F)|} + (n^k - n(n-1) \cdots (n-k+1))\delta \\ &= (1 + o(1))n^k p^{|E(F)|} \quad (n \rightarrow \infty) \end{aligned}$$

since

$$\lim_{n \rightarrow \infty} \frac{(n^k - n(n-1) \cdots (n-k+1))\delta}{n^k p^{|E(F)|}} = 0$$

for fixed $p, k, E(F)$ and $0 \leq \delta \leq 1$. □

3.2 Right & unweighted

Example 6. Let H be a point, then $\text{hom}(G, H) \neq 0$ iff G has no edge.

Let H be K_2 , then $\text{hom}(G, H) \neq 0$ iff G is bipartite.

Let H be K_3 , then $\text{hom}(G, H) \neq 0$ iff G is 3-colorable.

Example 7. (Independent Set) Let H be the graph on two nodes, with an edge connecting the two nodes and a loop at one of the nodes. Then $\text{hom}(G, H)$ is the number of independent sets of nodes in G .

Proof. The independent set is 1-1 corresponding to $\phi^{-1}(2)$. More precisely, given any independent set A , there is a unique $\phi : G \rightarrow H$ such that $A = \phi^{-1}(2)$. On the other hand, $\phi^{-1}(2)$ is an independent set for any given homomorphism $\phi : G \rightarrow H$. □

Note: If H has only two nodes $\{1, 2\}$, any $\phi : G \rightarrow H$ is uniquely decided by any fibre of ϕ . ($\phi^{-1}(1)$ or $\phi^{-1}(2)$) Or we can say such ϕ given a partition on G . In this sense there exists a 1-1 corresponding between the partitions and the homomorphisms where (V_1, V_2) and (V_2, V_1) are treated as two different partitions. In the above example, the part of partition corresponding to $\phi^{-1}(2)$ is exactly the independent set, which is implied by the fact that point 2 is loopless.

Example 8. (Colorings) Let H be K_t , then $\text{hom}(G, H) = \#$ of the colorings of the graph G with t colors.

Note: We can consider the homomorphisms that mapped into a fixed graph H as generalized colorings, called the H colorings. There are also other generalizations, such as circular colorings and fractional colorings.

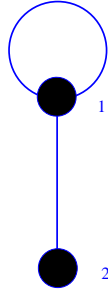


Figure 4: the target for Independent Set

3.3 Weighted examples

Example 9. (Maximum cut) Let H denote the looped complete graph on two nodes, weighted as follows: the non-loop edge has weight 2; all other edges and nodes have weight 1. Then for every simple graph G with n nodes,

$$\log_2 \text{hom}(G, H) - n \leq \text{MaxCut}(G) \leq \log_2 \text{hom}(G, H).$$

where $\text{MaxCut}(G)$ denotes the size of the maximum cut in G . So unless G is very sparse, $\log_2 \text{hom}(G, H)$ is a good approximation of the maximum cut in G .

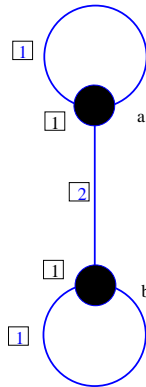


Figure 5: the target for Maximum Cut

Proof. If $\{V_1, V_2\}$ is a partition of $V(G)$, the set $E(V_1, V_2)$ of all edges of G crossing this partition is called a cut, whose size is denoted by $t_{\{V_1, V_2\}}$. From the discussion in Example 7, we know there is a 1-1 correspondence between the partitions and the homomorphisms. On the other hand, the correspondence between partitions and cuts is many-to-one.

Choose a cut of maximal size, denote by t_{max} , and a homomorphism ϕ_{max} (not unique) corresponding to this cut.

From the weight on H , we know:

$$hom(G, H) = \sum_{\phi: G \rightarrow H} hom_{\phi}(G, H) = \sum_{\phi: G \rightarrow H} 2^{t_{\phi}}$$

where t_{ϕ} is the cut size corresponding to the partition given by ϕ .

Then we have

$$2^{t_{max}} = hom_{\phi_{max}}(G, H) \leq hom(G, H),$$

which means

$$\text{MaxCut}(G) \leq \log_2 hom(G, H).$$

On the other hand,

$$hom(G, H) \leq 2^n hom_{\phi_{max}}(G, H) = 2^{n+t_{max}}$$

as there are total 2^n homomorphisms from G to H , which implies:

$$\log_2 hom(G, H) - n \leq \text{MaxCut}(G).$$

□

Example 10. (Partition functions of the Ising model) Let G be any simple graph, and let $T > 0$, $h \geq 0$, and J be three real parameters. Let H be the looped complete graph on two nodes, denoted by $+$ and $-$, weighted as follows: $\alpha_+ = e^{h/T}$, $\alpha_- = e^{-h/T}$, $\beta_{++} = \beta_{--}$, and $\beta_{+-}/\beta_{-+} = e^{2J/T}$. Then $hom(G, H)$ is the partition function of the Ising model on the graph G at temperature T with coupling J in external magnetic field h .

Proof. The proof will be left as an exercise to the reader. (Hint: the configurations is 1-1 corresponding to the homomorphisms via its fibres.)

□

3.4 As a language

- Chromatic Number:

$$\chi(G) = \min_k \{k \mid hom(G, K_n) \neq 0\}$$

- Clique Number:

$$\omega(G) = \max_n \{k \mid hom(K_n, G) \neq 0\}$$

- Odd girth:

$$og(G) = \min_l \{2l + 1 \mid hom(C_{2l+1}, G) \neq 0\}$$

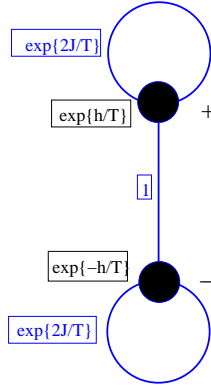


Figure 6: the target for partition function

4 A nontrivial Example

4.1 Nowhere-zero flows

Let Γ be a finite abelian group and let S be a subset of Γ s.t S is closed under inversion.

For any graph G , fix an orientation of the edges. An S -flow is an assignment of an element of S to each edge s.t for each node v , the sum of elements assigned to edges entering v is the same as the sum of elements assigned to edges leaving v .

Let $f(G)$ be the number of S -flows. Then this number is independent of the choice of orientation.

Let $S = \Gamma \setminus \{0\}$. Then such an S -flow is called nowhere-zero flow.

A Eulerian tour is an S -flow when $\Gamma = \mathbb{Z}_2$ and $S = \mathbb{Z}_2 \setminus \{0\}$.

4.2 The representation of flows number

Let H be the complete directed graph (with all loops) on $\hat{\Gamma}$. Let $\alpha_\chi \triangleq \frac{a}{|\Gamma|}$ for each $\chi \in \Gamma$, and let

$$\beta_{\chi, \chi'} \triangleq \sum_{s \in S} \chi^{-1}(s) \chi'(s),$$

for $\chi, \chi' \in \hat{\Gamma}$. Then $f(G)$ can be described as a homomorphism function [1].

Theorem 4.1. $f(G) = \text{hom}(G, H)$.

The proof of this theorem is rather technical and will be put in the appendix. Instead, we will present two examples: the first one is the H for the Eulerian characteristic function and the second is the H for the Nowhere-zero 4 flows.

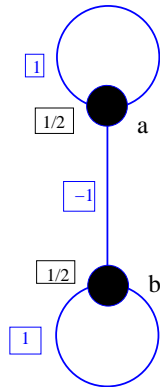


Figure 7: The H for Eulerian

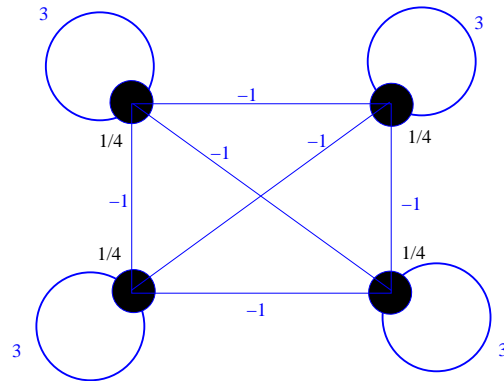


Figure 8: the H for Nowhere-zero 4 flow

But we can sketch a proof for the special case for Eulerian tour, which in fact gives a new proof of the Eulerian theorem.

Proof. • 1: If G has Eulerian tour, then it can only have just one such tour.

- 2: If all vertices have even degree, then the size of any cut set is even.
- 3: If some vertices have odd degree, then the number of cut sets with even size equal to that of odd size.
 - The number of vertices having odd degree is even.
 - The parity of the cut size is totally decided by the parity of the vertices having odd degree.
 - Consider all possible partitions of $2k$ objects, the number of odd partition is equal to the even partition, which can be proven by induction.
- 4: Direct calculation gives the answer.

□

5 Some elementary properties

If F is the disjoint union of two graphs F_1 and F_2 , then

$$\text{hom}(F, G) = \text{hom}(F_1, G)\text{hom}(F_2, G).$$

If F is connected and G is the disjoint union of two graphs G_1 and G_2 , then

$$\text{hom}(F, G) = \text{hom}(F, G_1) + \text{hom}(F, G_2).$$

Thus in a sense it's enough to study homomorphisms between connected graphs.

For two simple graphs G_1, G_2 , their *categorical product* $G_1 \times G_2$ is defined to be a graph with vertices set $V(G_1) \times V(G_2)$, in which (i_1, j_1) is connected to (i_1, j_2) iff $(i_1, i_2) \in E(G_1)$ and $(j_1, j_2) \in E(G_2)$. For this product, we have:

$$\text{hom}(F, G_1 \times G_2) = \text{hom}(F, G_1) \cdot \text{hom}(F, G_2)$$

6 Conclusion

Today we are focused on some examples of graph parameters which can be represented by counting the graph homomorphisms. A natural question would be: what are the parameters that are unable to have such representation? Surprisingly, the authors of [2] obtain some exact conditions to character such parameters, which would likely to be the topic of the next talk of this series. Or the reader can consult Section 3 in [2].

7 Appendix

7.1 A concrete example

In this subsection we will consider a concrete graph G , which is obtained by removing an edge from K_4 .

For graph G , we have

$$A(G) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \quad A^2(G) = \begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

On the other hand, we have

$$\det(\lambda I - A) = \lambda^4 - 5\lambda^2 - 4\lambda = \lambda(\lambda + 1)(\lambda^2 - \lambda - 4),$$

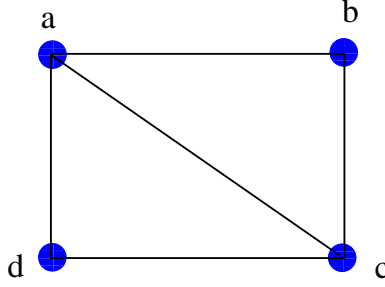


Figure 9: Welsh Graph

from which we can obtain the eigenvalues of $A(G)$:

$$\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = \frac{1}{2}(1 + \sqrt{17}), \lambda_4 = \frac{1}{2}(1 - \sqrt{17})$$

Now we investigate which properties can we get by counting the homomorphisms from left and right. Firstly, from left:

- $hom(K_1, G) = |V(G)| = 4$.
- $hom(K_2, G) = 2|E(G)| = 10 = \sum_{i,j} a_{ij}$ where $K_2 = S_2 = P_2$
- $hom(K_3, G) = 12 = 6 \times$ the number of triangles in G .
- $hom(P_3, G) = 26 = \sum_{i,j} ({}^2a_{ij})$.
- $hom(S_3, G) = \sum_1^4 d_i^2 = 26$
- $hom(C_2, G) = 10 = \sum_i ({}^2a_{ii}) = \sum_i \lambda_i^2$.

For the right target, we use Independent Set to denote the graph H we constructed as the target for the independent set and the same convenience for others. For the later use, we list the cut sizes and the partitions in the following table. (Note: this is unordered partition, in homomorphism we should use the ordered one).

partition	{a}	{a,b}	{a,c}	{a,d}	{a,b,c}	{a,b,d}	{a,c,d}	{a,b,c,d}
cut size	3	3	4	3	2	2	3	0

- $hom(G, K_2) = 0$, which means G is not bipartite.
- $hom(G, K_3) = 6$, the number of 3 colors.
- $hom(G, \text{Independent Set}) = 6$, they are $\{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{b, c\}\}$.
- $hom(G, \text{Maximum Cut}) = 2 \sum 2^{t_i} = 98$ while $\log_2 98 \approx 6..$
- $hom(G, \text{Eulerian}) = 2 \sum (\frac{1}{2})^4 (-1)^{t_i} = 0$.

7.2 The Ising Model

Let $G = (V, E)$ be a finite graph, and call $\Omega = \{-1, +1\}^V$ the state space, with elements $\sigma = \{\sigma_x\}_{x \in V}$. The variable $\sigma_x \in \{-1, +1\}$ is called the spin at vertex x . This is a *spin system*.

There is an *energy function* defined on Ω . For the Ising model this function is defined as:

$$H(\sigma) = -J \sum_{(x,y) \in E} \sigma_x \sigma_y - \sum_{x \in V} h \sigma_x.$$

where J is a real constant, the *interaction strength*, and $h \in \mathbb{R}$, decided by an *external magnetic field*.

Now we introduce the *inverse temperature* parameter $\beta \sim \frac{1}{T}$, and consider the following probability measure on Ω :

$$\mu(\sigma) = \frac{e^{-\beta H(\sigma)}}{Z_\beta}, \quad Z_\beta = \sum_{\sigma \in \Omega} e^{-\beta H(\sigma)}. \quad (4)$$

This Z_β is called the *partition function*, and, as usually generating functions do, contains basically all information about the systems.

7.3 Group Characters

A *character* χ of Γ is a homomorphism $\Gamma \rightarrow \mathbb{S}^1$ where \mathbb{S}^1 is the multiplicative group of complex numbers of modulus 1. The *unit character* χ_0 is the character which assigns 1 to every element in Γ .

We list a few facts about characters:

- A function $\chi : \Gamma \rightarrow \mathbb{C}$ is a character iff it satisfies $\chi(a + b) = \chi(a) + \chi(b)$, $a, b \in \Gamma$ since Γ is finite.
- The set of all characters of Γ form a group $\hat{\Gamma}$, called the dual group of Γ .
- $\hat{\Gamma} \cong \Gamma$.

Proposition 7.1.

$$\sum_{\chi \in \hat{\Gamma}} \chi(a) = \begin{cases} n & \text{if } a = 0 \\ 0 & \text{otherwise} \end{cases}$$

7.4 The proof of Theorem 4.1

Proof. Let $n = |V(G)|$ and $m = |\Gamma|$.

For any coloring $\psi : E(G) \rightarrow S$ and node $v \in V(G)$, let

$$\partial_\psi(v) = \sum_{u \in V(G)} \sum_{uv \in E(G)} \psi(uv) - \sum_{u \in V(G)} \sum_{vu \in E(G)} \psi(vu).$$

So ψ is an S -flow iff $\partial_\psi = 0$.

Consider the expression

$$A = \sum_{\psi: E(G) \rightarrow S} \prod_{v \in V(G)} \sum_{\chi \in \hat{\Gamma}} \chi(\partial_\psi(v)). \quad (5)$$

From Proposition 7.1, the summation over χ is 0 unless $\partial_\psi(v) = 0$, in which case it is m . So the product over $v \in V(G)$ is 0 unless ψ is an S -flow, in which case it is m^n . Therefore $A \cdot m^{-n}$ counts S -flows.

On the other hand, we can expand the product over $v \in V(G)$; this step looks like:

$$\begin{aligned} & (\chi_0(\partial_\psi(v_1)) + \chi_1(\partial_\psi(v_1)) + \cdots + \chi_{m-1}(\partial_\psi(v_1))) \quad \times \\ & (\chi_0(\partial_\psi(v_2)) + \chi_1(\partial_\psi(v_2)) + \cdots + \chi_{m-1}(\partial_\psi(v_2))) \quad \times \\ & \quad \quad \quad \dots \quad \quad \quad \times \\ & (\chi_0(\partial_\psi(v_n)) + \chi_1(\partial_\psi(v_n)) + \cdots + \chi_{m-1}(\partial_\psi(v_n))) \\ = & \sum_{\phi} (\chi_{\phi(v_1)}(\partial_\psi(v_1))) (\chi_{\phi(v_2)}(\partial_\psi(v_2))) \cdots (\chi_{\phi(v_n)}(\partial_\psi(v_n))) \end{aligned}$$

Each term in the sum is corresponding to a choice of a character χ_{ϕ_v} for each v . Denote χ_{ϕ_v} by ϕ_v and so we get

$$A = \sum_{\psi: E(G) \rightarrow S} \sum_{\phi: V(G) \rightarrow \hat{\Gamma}} \prod_{v \in V(G)} \phi_v(\partial_\psi(v)).$$

Here (using that ϕ_v is a character)

$$\phi_v(\partial_\psi(v)) = \prod_{u \in V(G)} \prod_{uv \in E(G)} \phi_v(\psi(uv)) \prod_{u \in V(G)} \prod_{vu \in E(G)} \phi_v(\psi(vu))^{-1}$$

So we get that

$$A = \sum_{\psi: E(G) \rightarrow S} \sum_{\phi: V(G) \rightarrow \hat{\Gamma}} \prod_{uv \in E(G)} \phi_v(\psi(uv)) \phi_u(\psi(uv))^{-1}.$$

Interchanging the summation, we have:

$$\begin{aligned}
A &= \sum_{\phi:V(G)\rightarrow\hat{\Gamma}} \sum_{\psi:E(G)\rightarrow S} \prod_{uv\in E(G)} \phi_v(\psi(uv)) \phi_u(\psi(uv))^{-1} \\
&= \sum_{\phi:V(G)\rightarrow\hat{\Gamma}} \prod_{uv\in E(G)} \sum_{s\in S} \phi_v(s)\phi_u(s)^{-1} \\
&= \sum_{\phi:V(G)\rightarrow\hat{\Gamma}} \prod_{uv\in E(G)} \beta_{\phi(u),\phi(v)} \\
&= \sum_{\phi:V(G)\rightarrow\hat{\Gamma}} m^n \text{hom}_{\phi}(G, H) \\
&= m^n \text{hom}(G, H).
\end{aligned}$$

where in the last two steps we use the following fact:

$$\begin{aligned}
\text{hom}(G, H) &= \sum_{\phi:V(G)\rightarrow V(H)} (\alpha_{\phi})(\text{hom}_{\phi}(G, H)) \\
&= \sum_{\phi:V(G)\rightarrow V(H)} \left(\prod_{u\in V(G)} [\alpha_{\phi(u)}(H)] \right) \left(\prod_{uv\in E(G)} [\beta_{\phi(u)\phi(v)}(H)] \right) \\
&= \sum_{\phi:V(G)\rightarrow V(H)} \left(\frac{1}{m^n} \right) \left(\prod_{uv\in E(G)} [\beta_{\phi(u)\phi(v)}(H)] \right). \\
&= \sum_{\phi:V(G)\rightarrow V(H)} \left(\frac{1}{m^n} \right) \text{hom}_{\phi}(G, H).
\end{aligned}$$

Therefore, we have:

$$f(G) = m^{-n} A = \text{hom}(G, H)$$

which completes our proof. \square

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- [1] M.Freedman, L.Lovasz, A. Schrijver, Reflection positivity, rank connectivity, and homomorphism of graphs.
- [2] Ch.Borgs, J.Chayes, L.Lovasz, V.T.Sos and K.Vesztergombi, counting graph homomorphisms.