Solitaire SET

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The SET game

SET, marketed by Set Enterprises, Inc., is a card game for 1 or more players. We give here a brief description. Alternatively, details are available in [1], or you can acquire the game and read the instructions.

The are 81 cards with various colourful designs on them. They represent points (a, b, c, d), $a, b, c, d \in \{0, 1, 2\}$, in $\mathrm{AG}(4, 3)$, the affine geometry of dimension 4 over $\mathbb{Z}_3$. A line, which the game's inventor calls a *SET*, consists of three cards that sum to $(0, 0, 0, 0)$ modulo 3. For notational convenience, we follow the convention in [1]: SET is the game; a *SET* is a line in $\mathrm{AG}(4, 3)$.

The general idea is that in a competitive environment players attempt to identify *SET*s. The rules are simple.

Play begins with the dealer dealing 12 cards from the deck face-up on to the table. Thereafter, players compete with each other to find *SET*s. There is no taking of turns. If you think you have identified a *SET*, you call "Set!" and you are then obliged—with a suitable forfeit if you fail—to remove a *SET* from the table, which if necessary is replenished by the dealer to bring the number back up to 12. If there is no *SET* on the table, the dealer deals three more cards if possible. The following array shows a typical state of the game.

(2,1,0,2)	$(1,2,0,1)$	$(0,1,2,1)$	$(2,2,0,2)$	$(2,2,1,2)$	$(2,0,1,1)$
$(0,0,1,2)$	**(1,0,2,1)**	**(0,2,1,0)**	$(0,2,0,0)$	$(1,0,1,0)$	$(1,1,1,2)$

There are eight *SET*s on the table, one which is highlighted. See if you can identify the other seven.

A game ends when there are no more cards to deal and no more *SET*s to take. The winner is the person with the most *SET*s.

Our encoding of the SET cards is the same as in the book, [1, Table 8.1]. A card is represented by the vector (number, colour, filling, shape) with the elements coded as:

number:	$3 \to 0$,	$1 \to 1$,	$2 \to 2$;
colour:	green $\to 0$,	purple $\to 1$,	red $\to 2$;
filling:	empty $\to 0$,	hatched $\to 1$,	solid $\to 2$;
shape:	diamond $\to 0$,	oval $\to 1$,	squiggle $\to 2$.

For instance, $(0, 1, 1, 0)$ is the card depicting 3 purple hatched diamonds.

The solitaire game

We are interested in the solitaire version of SET. There is one player, yourself. Your aim is to achieve a perfect score, 27 *SET*s; that is, the game ends with no cards left on the table—all clear. The front cover shows such a deal, where at each stage while there are undealt cards, if there are two or more *SET*s on the table, you take the first in the ordering defined by (1) on page 3. However, there is no guarantee that a game can finish all clear. For an example of one that cannot, see [1, Section 10.7].

At any stage of a solitaire game the way ahead is clear if the table is *SET*-free or contains just one *SET*. The fundamental problem is how to make good choices on those occasions where there are two or more *SET*s on the table.

According to the book, a run of 100000000 games where *SET*s are chosen at random produced a mere 1.22 per cent all clear, [1, Table 10.4]. We claim that we can increase this hit rate to something better than 10 per cent. Before explaining our algorithm, we must be very precise about the way the game is played. The rules are as in the usual many-player version, but we emphasise the following clarifications.

(i) At each stage of the game, you must either *prove* there is no *SET* on the table,[1] or take exactly one *SET* from the table, which is immediately replenished to 12 cards if necessary and possible. Passing is forbidden if the table contains a *SET*. This rule is vital. Without it the solitaire game becomes pointless. Just deal all 81 cards (pretending not to notice any *SET*s) and then partition them into 27 *SET*s.

(ii) Obviously you can see[2] the cards on the table.

(iii) You can see all the *SET*s taken from the table.

(iv) From (ii) and (iii), you can deduce the undealt cards. However, you are not allowed to know how they are ordered—that would be cheating.

(v) You can see all possible *SET*s formed from the cards on the table.

(vi) You can compute all possible partitions into *SET*s of the undealt cards together with the cards on the table.

(vii) You must make your choice within a finite time. We suggest 3 years for a human, 3 billion clock-ticks for an electronic computer.

[1] *The table* refers to the playing area on to which the cards are dealt. To avoid possible confusion, you should place all *SET*s taken and all undealt cards on a grand piano.

[2] Here and elsewhere, *see* means determine the points in $AG(4,3)$ that are represented by the cards. (This might not be possible if the cards in your view are face down.)

Definitions

To avoid random processes in our proposed algorithm, we need to specify how to order cards and *SET*s. If $x = (x_1, x_2, x_3, x_4)$, $x_1, x_2, x_3, x_4 \in \{0, 1, 2\}$, is a card in its vector form, we define

$$N(x) \;=\; N((x_1, x_2, x_3, x_4)) \;=\; 27x_1 + 9x_2 + 3x_3 + x_4.$$

Thus N maps the SET deck on to $\{0, 1, \ldots, 80\}$ and defines an ordering: if x and y are cards, then

$$x < y \;\text{ if }\; N(x) < N(y).$$

The extension to *SET*s is as expected. If $\{a, b, c\}$, $a < b < c$, and $\{d, e, f\}$, $d < e < f$, are distinct *SET*s, then

$$\{a, b, c\} < \{d, e, f\} \;\text{ if }\; a < d \text{ or } a = d \text{ and } b < e. \tag{1}$$

If x and y are distinct cards, we denote by $x * y$ the unique card that makes a *SET*, $\{x, y, x * y\}$. Also we say that

the pair $\{x, y\}$ is *dead* if $x * y$ is amongst the *SET*s taken;

the pair $\{x, y\}$ is *live* if $x * y$ is in the undealt part of the deck.

Be aware that not live is not the same as dead.

The basic strategy

Proceed as usual if there is no *SET* or exactly one *SET* on the table. However, when there are two or more *SET*s on the table you must make a choice.

Let $\mathcal{X}$ be the collection of all candidate *SET*s that would maximize the number of live pairs on the table if the *SET* were taken.

If $\mathcal{X}$ consists of just one *SET*, choose it. Otherwise select from $\mathcal{X}$ those *SET*s that would minimize the number of dead pairs on the table if the *SET* were taken. If a tie-break is still needed, choose the first *SET* according to the ordering defined by (1). Remove the chosen *SET*.

The idea is that live pairs are good and worth preserving for possible future *SET*s. On the other hand, dead pairs are bad and we should try to get rid of as many as possible when we choose our *SET*.

The general strategy $S(d)$, $d \geq 0$

We find that the basic strategy works well; the proportion of games ending all clear is increased from 1.22% to more than 5.0%. However, it turns out that we can do significantly better.

For the general strategy, proceed with the basic strategy until there are d undealt cards, where d is a not too large multiple of 3. Upon arriving at this stage in the game we may assume that either there are at least 12 cards on the table or there are no undealt cards.

Construct all partitions into SETs of the cards on the table together with the undealt cards.

If there are no partitions, the game must end not all clear.

Otherwise we argue as follows.

If $d = 0$, then the table is neatly partitioned into SETs, all available for taking. The game ends all clear.

Next, assume $d = 3$. Then there is a partition of at least 15 cards (3 undealt and at least 12 on the table) into at least 5 SETs. Of these, at least 2 are on the table since the SETs are mutually disjoint and therefore at most 3 can include undealt cards. Removing one of the SETs from the table reduces the situation to $d = 0$, above, and the game ends all clear.

Finally, assume $d \geq 6$.

Suppose one of the partitions, $\mathcal{P}$ say, has $d/3 - 1$ of its SETs on the table. Now it is possible to choose a SET from $\mathcal{P}$ at the next $d/3 - 1$ turns to leave 3 undealt cards and at least 12 cards on the table. Then case $d = 3$ applies, and the game ends all clear.

On the other hand, if there is no such partition $\mathcal{P}$, we do the basic strategy followed immediately by general strategy $S(d - 3)$.

Short cuts and refinements

In each of the following cases we assume there exists a partition into SETs of the undealt cards together with the cards on the table.

(i) If there are 0 or 3 undealt cards, the game ends all clear—as already explained.

(ii) If there are 6 undealt cards and at least 15 cards on the table, the game ends all clear. The partition contains at least 7 SETs of which one will be on the table since at most 6 can contain undealt cards.

(iii) When there are 6 undealt cards and the table is SET-free, the game ends all clear. Since 3 cards must be dealt, the undealt card count is reduced to 3 for (i) to apply.

(iv) When there are 9 undealt cards and the table is SET-free, the game ends all clear. Since 3 cards must be dealt, the undealt card count goes to 6 and the table card count goes to at least 15. Now (ii) applies.

Results

The following shows results for general strategy $S(d)$ augmented with (i)–(iv) of Short cuts and refinements.

d	games	all clear	%
0	200000	14082	7.0
3	200000	14948	7.5
6	200000	18394	9.2
9	500000	52793	10.6
12	500000	53196	10.6

The recommended strategy is $S(9)$ for an all-clear rate of about 10.6 per cent. (Recall that choosing *SET*s at random achieves only about 1.22 per cent.) We consider the apparent small improvement offered by $S(12)$ to be insignificant. Moreover, we find that the cost of implementing $S(d)$ increases considerably when d goes from 9 to 12.

Reference

[1] Liz McMahon, Gary Gordon, Hannah Gordon and Rebecca Gordon, *The Joy of SET*, Princeton University Press, 2017; reviewed in M500 **276**.