Hall's marriage theorem

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We present a proof of Hall's marriage theorem inspired by Chapter 21 of [1], which was based on [2].

For the theorem and its proof, we assume that n is a positive integer, X is a set of at least n elements, and

$$\mathcal{T} \ = \ (T_1, T_2, \ldots, T_n)$$

is a collection of subsets of X.

Definition *Hall's condition* holds for a collection $\mathcal{A}$ of $m \geq 1$ finite sets if for $k = 1, \ldots, m$, the union of any k sets of $\mathcal{A}$ contains at least k elements.

Definition A *system of distinct representatives*, *SDR*, for a collection of $m \geq 1$ finite sets $(A_1, A_2, \ldots, A_m)$ is a vector $(s_1, s_2, \ldots, s_m)$ of distinct elements such that $s_i \in A_i$ for $i = 1, 2, \ldots, m$.

Definition For integer c, $0 \leq c \leq n$, a *critical collection* for $\mathcal{T}$ is a collection $T_{i_1}, T_{i_2}, \ldots, T_{i_c}$, with distinct indices $\{i_1, i_2, \ldots, i_c\}$, that covers precisely c elements of X.

Repetitions are allowed in $\mathcal{T}$ and its critical collections. For example,

$$\mathcal{T} \ = \ (\{1,6,7\}, \{2,4,6,7\}, \{1,7\}, \{3,5\}, \{1,3,7\}, \{3,5\})$$

has the critical collection $(\{1,7\}, \{3,5\}, \{1,3,7\}, \{3,5\})$.

If $\mathcal{T}$ satisfies Hall's condition and $|\mathcal{T}| = |X|$, then both $\{\}$ and $\mathcal{T}$ are critical collections. These are a bit of a nuisance, and so we make a further definition involving criticality.

Definition A *proper critical collection* for $\mathcal{T}$ is a critical collection of size at least 1 and at most $n - 1$.

Theorem 1. *Suppose Hall's condition holds for $\mathcal{T}$; i.e. for $k = 1, 2, \ldots, n$, the union of any k sets of $\mathcal{T}$ contains at least k elements of X.*

Then there exists an SDR for $\mathcal{T}$; i.e. there exists a vector $(s_1, s_2, \ldots, s_n)$ of distinct elements of X such that $s_i \in T_i$ for $i = 1,\ 2,\ \ldots,\ n$.

The following interpretation is familiar. The elements of X are boys. The indices i of the T_i are girls. The elements of T_i are the boyfriends of girl i. The existence of an SDR for $\mathcal{T}$ means that each girl can marry one of her boyfriends. The system is not symmetric; we are not trying to find suitable wives for the boys.

Proof. We do it by induction. The theorem is clearly true for $n = 1$.

Assume there exists an SDR for any collection of finite sets $\mathcal{A}$ with $|\mathcal{A}| \in \{1, 2, \ldots, n-1\}$ that satisfies Hall's condition.

Assume Hall's condition holds for $\mathcal{T}$.

There are two cases.

Case 1: There is no proper critical collection.

Let w be an element of T_n. Remove T_n and w from $\mathcal{T}$ to obtain

$$\mathcal{U} \;=\; (U_1,\ U_2,\ \ldots,\ U_{n-1}) \;=\; (T_1 \setminus \{w\},\ T_2 \setminus \{w\},\ \ \ldots,\ T_{n-1} \setminus \{w\}).$$

Since there is no proper critical collection for $\mathcal{T}$, any k sets of $\mathcal{T} \setminus \{T_n\}$ must cover at least $k+1$ elements. Hence any k sets of $\mathcal{U}$ must cover at least k elements.

Therefore Hall's condition holds for $\mathcal{U}$.

By the induction hypothesis, there exists an SDR $(s_1, s_2, \ldots, s_{n-1})$ for $\mathcal{U}$. Augmenting this with $s_n = w$ gives an SDR for $\mathcal{T}$.

Example

$$\begin{aligned}
\mathcal{T} \;&=\; (\{4,9\}, \{1,4,7,10\}, \{2,3\}, \{3,6,8,10\}, \{3,6,7\}, \{1,5,8\},\\
&\qquad \{2,3,4,5,9,10\}, \{2,5,10\}, \{5,8\}, \{9,10\}),\\
w \;&=\; 9,\\
\mathcal{U} \;&=\; (\{4\}, \{1,4,7,10\}, \{2,3\}, \{3,6,8,10\}, \{3,6,7\}, \{1,5,8\},\\
&\qquad \{2,3,4,5,10\}, \{2,5,10\}, \{5,8\}),\\
\text{SDR} \;&:\; (4,1,3,6,7,5,2,10,8).
\end{aligned}$$

Case 2: There exists a proper critical collection.

After relabelling if necessary, assume there exists a proper critical collection of size c, $1 \leq c \leq n-1$,

$$\mathcal{C} \;=\; (T_1,\; T_2,\; \ldots,\; T_c),$$

and let

$$Y \;=\; T_1 \cup T_2 \cup \cdots \cup T_c \;\subset\; X.$$

By the induction hypothesis, there exists an SDR $(s_1, s_2, \ldots, s_c)$ for $\mathcal{C}$, with $\{s_1, s_2, \ldots, s_c\} = Y$.

Consider now the remaining sets, $\mathcal{D} = (T_{c+1}, T_{c+2}, \ldots, T_n)$. Take any collection $\mathcal{K}$ of k sets of $\mathcal{D}$.

By Hall's condition, the k sets of $\mathcal{K}$ together with the c sets of $\mathcal{C}$ cover at least $k + c$ elements of X. Moreover, at least k of these elements are not in Y. Therefore Hall's condition holds for

$$\mathcal{E} \;=\; (T_{c+1} \setminus Y,\; T_{c+2} \setminus Y,\; \ldots,\; T_n \setminus Y).$$

Again by the induction hypothesis, there exists an SDR $(s_{c+1}, s_{c+2}, \ldots, s_n)$ for $\mathcal{E}$, and hence for $\mathcal{D}$ with elements not in Y.

Combining the SDR for $\mathcal{C}$ with the SDR for $\mathcal{E}$ gives an SDR for $\mathcal{T}$. $\square$

Example

$$\begin{aligned}
\mathcal{T} &= (\{2,8,9\},\{4,6\},\{1,4\},\{6,8\},\{1,8,9\},\{2,4,6,7\},\{2,7,9\},\\
&\quad\;\{2,4,5,8,9,10\},\{2,7,10\},\{2,3,9,10\}),\\
\mathcal{C} &= (\{2,8,9\},\{4,6\},\{1,4\},\{6,8\},\{1,8,9\},\{2,4,6,7\},\{2,7,9\}),\\
Y &= \{1,2,4,6,7,8,9\},\\
\text{SDR} &: (2,4,1,6,8,7,9),\\
\mathcal{D} &= (\{2,4,5,8,9,10\},\{2,7,10\},\{2,3,9,10\}),\\
\mathcal{E} &= (\{5,10\},\{10\},\{3,10\}),\\
\text{SDR} &: (5,10,3).
\end{aligned}$$

An application of the proof of Hall's marriage theorem. Identify proper critical collections to solve this sudoku puzzle.

8	3679	**1**	45679	2456	23679	456	24567	2456
24567	679	24579	1456789	124568	126789	14568	**3**	124568
234567	367	2457	145678	124568	123678	**9**	124567	124568
1246	168	248	1689	1268	**5**	**7**	1469	134689
9	**5**	478	1678	**3**	1678	**2**	146	1468
1267	1678	**3**	16789	1268	**4**	1568	1569	15689
135	139	59	1456	**7**	16	13456	**8**	1234569
1357	**4**	**6**	**2**	**9**	18	135	15	135
15	**2**	589	**3**	14568	168	1456	14569	**7**

n=21 30 81 k=8 i=0 d=0 harder

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*8.1.............3.......9.......57..95..3.2....3..4.......7..8..4629.....2.3....7/
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References

[1] Martin Aigner and Günter Ziegler, *Proofs from THE BOOK*, Springer, 1999.

[2] Paul Halmos and Herbert Vaughan, The marriage problem, *Amer. J. Math.* **72** (1950), 214–215.