Bricks

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Given a supply of $1 \times 2 \times 6$ bricks, try packing 42 of them into an $8 \times 8 \times 8$ cube, a task which should not give you too much trouble. Or maybe you can pack 28 of these things into a $7 \times 7 \times 7$ cube—or prove that it cannot be done. More generally, what we are really after is the answer to the question:

> *What is the maximum number of* $1 \times 2 \times 6$ *bricks that you can pack into an* $n \times n \times n$ *cube,* $n = 4, 5, 6, \ldots$*?*

In the table we give some upper and lower bounds. Observe that when n is even the maximum packing number is determined exactly—the upper and lower bounds are the same.

n	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$\lfloor n^3/12 \rfloor$	5	10	18	28	42	60	83	110	144	183	228	281	341	409	486	571
unused	4	5	0	7	8	9	4	11	0	1	8	3	4	5	0	7
upper bound	0	2	18	28	42	60	82	110	144	183	228	280	340	409	486	571
unused	64	101	0	7	8	9	16	11	0	1	8	15	16	5	0	7
lower bound	0	2	18	27	42	57	82	108	144	180	228	276	340	405	486	567
unused	64	101	0	19	8	45	16	35	0	37	8	63	16	53	0	55

The 'unused' lines indicate how many holes are left unfilled by the possibly hypothetical packing. The lower bounds are obtained as follows.

$\boldsymbol{n = 6k}$ Left for the reader.

$\boldsymbol{n = 6k + 2}$ Lay $3k^2 + 2k$ bricks flat on an $n \times n$ square to leave a 2×2 hole in a corner. Pile n of these structures vertically and drop $2k$ bricks into the $2 \times 2 \times n$ hole:

$$n(3k^2 + 2k) + 2k \;=\; \frac{n^3 - 8}{12} \;=\; \left\lfloor \frac{n^3}{12} \right\rfloor \quad \text{bricks,}$$

best possible.

$\boldsymbol{n = 4}$ Left for the reader.

$\boldsymbol{n = 6k + 4 \geq 10}$ Lay $8k$ bricks flat to make a wall 4 units thick that encloses an $(n-8) \times (n-8)$ square region. If $n > 10$, apply the first part of the $6(k-1)+2$ construction to the region. Thus an $n \times n$ square is covered except for a 2×2 hole. Pile n of these structures vertically and drop $2k$

bricks into the $2 \times 2 \times n$ hole:

$$\frac{n(n^2-4)}{12} + 2k \;=\; \frac{n^3-16}{12} \;=\; \left\lfloor \frac{n^3}{12} \right\rfloor - 1 \;\text{ bricks,}$$

leaving a volume of 16 unoccupied.

Since 16 exceeds the volume of a brick by 4, it is tempting to suggest that perhaps there is some smart arrangement which accommodates one extra. However, no such packing exists; $\lfloor n^3/12 \rfloor - 1$ is best possible, as we shall prove in Theorem 1, below.

$\boldsymbol{n = 5}$ See Problem 315.1 on page 10.

Odd $\boldsymbol{n \geq 7}$ Take the arrangement for the $(n-1)\times(n-1)\times(n-1)$ cube and clad three mutually orthogonal faces with as many bricks as possible. Thus, for example, $18 + 3 \cdot 3 = 27$ for $n = 7$, and $42 + 3 \cdot 5 = 57$ for $n = 9$.

In some cases we can improve on this construction.

$\boldsymbol{n = 11}$ Use 20 bricks to build a wall 2 units high and 5 units thick that encloses a $1 \times 1 \times 2$ hole. Pile five of these structures vertically and lay eight bricks on top: $5 \cdot 20 + 8 = 108$ bricks.

$\boldsymbol{n = 15}$ See Theorem 2, below.

$\boldsymbol{n = 17}$ Use 40 bricks to build a wall 2 units high and 5 units thick that encloses a $7 \times 7 \times 2$ hole into which place eight more bricks. Pile eight of these structures vertically and lay 21 bricks on top: $8 \cdot 48 + 21 = 405$ bricks.

The reader is invited to reduce the gaps between the lower and upper bounds given for odd n in the table.

Also, we would be very interested in the smallest k for which you can put $18k^3 + 9k^2 + 1$ bricks in a $(6k+1) \times (6k+1) \times (6k+1)$ cube. To show that this is a sensible request, put $k = 10$, say. The construction described above involving the cladding of three faces of a $60 \times 60 \times 60$ cube uses 18900 bricks and leaves an unused volume of 181, quite a lot more than sufficient for one extra brick.

If it helps, there is a child's toy marketed under the name Tumbling Tower that consists of fifty-four $1 \times 2 \times 6$ bricks neatly fashioned out of wood and packaged in a $6 \times 6 \times 18$ cardboard and plastic box. Although we suspect its intended purpose is not to investigate the filling of cubical bins with small cuboids, we did actually find it useful.

Finally, we have the following results, which show that the trivial upper bound cannot be attained for $n \equiv 4 \pmod 6$ and $n \equiv 3 \pmod{12}$.

Theorem 1 *Let k be a positive integer and let $n = 6k + 4$. The maximum number of $1 \times 2 \times 6$ bricks that you can pack into an $n \times n \times n$ cube is $\lfloor n^3/12 \rfloor - 1$.*

Proof We have already shown how to pack the $n \times n \times n$ cube with $\lfloor n^3/12 \rfloor - 1$ bricks. So we only need to prove that $\lfloor n^3/12 \rfloor$ is impossible.

We think it is safe to assume that a brick in the packing must be orientated so that each of its six faces lies on one of the $3(n + 1)$ grid-planes that partition the cube into n^3 subcubes.

The proof involves polynomials in three complex variables. It might be helpful to follow the argument with $n = 10$.

Let the cube occupy $[0, n] \times [0, n] \times [0, n]$ in Euclidean 3-dimensional space, and suppose it is packed with $\lfloor n^3/12 \rfloor$ bricks to leave four of its subcubes unoccupied. The location of a compact set of points S is the $(a, b, c) \in S$ that minimizes each of the coordinates a, b and c.

Associate a subcube located at (a, b, c) with the polynomial $x^a y^b z^c$, where x, y and z are complex variables. Then the sum of the n^3 polynomials associated with the n^3 subcubes is

$$C(x, y, z) = \sum_{a=0}^{n-1} \sum_{b=0}^{n-1} \sum_{c=0}^{n-1} x^a y^b z^c.$$

A brick located at (a, b, c) is represented by a polynomial

$$x^a y^b z^c \, B_j(x, y, z),$$

where $B_j(x, y, z)$, $j \in \{1, 2, \ldots, 6\}$ is one of

$$\begin{aligned}
B_1(x, y, z) &= (1 + x + x^2 + x^3 + x^4 + x^5)(1 + y), \\
B_2(x, y, z) &= (1 + x + x^2 + x^3 + x^4 + x^5)(1 + z), \\
B_3(x, y, z) &= (1 + y + y^2 + y^3 + y^4 + y^5)(1 + x), \\
B_4(x, y, z) &= (1 + y + y^2 + y^3 + y^4 + y^5)(1 + z), \\
B_5(x, y, z) &= (1 + z + z^2 + z^3 + z^4 + z^5)(1 + x), \\
B_6(x, y, z) &= (1 + z + z^2 + z^3 + z^4 + z^5)(1 + y),
\end{aligned}$$

depending on the orientation of the brick in the cube. For example, $B_6(x, y, z)$ corresponds to a brick standing upright with its side of length 2 in the direction of the y-axis. The polynomial represents the 12 subcubes occupied by the brick, on the assumption that it is located at $(0, 0, 0)$. Shifting the brick to (a, b, c) corresponds to multiplying $B_6(x, y, z)$ by $x^a y^b z^c$.

The sum of the polynomials associated with the bricks in the packing is

$$B(x,y,z) \;=\; \sum_{j=1}^{6} P_j(x,y,z) B_j(x,y,z)$$

for some polynomials $P_1(x,y,z)$, $P_2(x,y,z)$, ..., $P_6(x,y,z)$. Thus $B(x,y,z)$ is the sum of $\lfloor n^3/12 \rfloor$ expressions of the form $x^a y^b z^c B_j(x,y,z)$ for various (a,b,c) and various $j \in \{1,2,\ldots,6\}$.

But there are also four points corresponding to the unused subcubes. Assuming they occur at distinct locations

$$(a_1,b_1,c_1), \quad (a_2,b_2,c_2), \quad (a_3,b_3,c_3), \quad (a_4,b_4,c_4),$$

the sum of their associated polynomials is

$$U(x,y,z) \;=\; \sum_{h=1}^{4} x^{a_h} y^{b_h} z^{c_h}.$$

Note that $U(x,y,z)$ depends on the parameters a_h, b_h, c_h.

For the assumed packing, there must exist polynomials $P_1(x,y,z)$, $P_2(x,y,z)$, ..., $P_6(x,y,z)$ and point coordinates $a_h, b_h, c_h \in \{0,1,\ldots,n-1\}$, $h = 1, 2, 3, 4$, such that

$$C(x,y,z) \;=\; B(x,y,z) + U(x,y,z) \quad \text{for all complex } x,y,z. \tag{1}$$

Now for the clever part. Put

$$x \;=\; y \;=\; z \;=\; \rho \;=\; \frac{1}{2} + \frac{\sqrt{3}\,i}{2},$$

a primitive 6th root of 1. Then, observing that $1+\rho+\rho^2+\rho^3+\rho^4+\rho^5 = 0$, we have

$$\begin{aligned} C(\rho,\rho,\rho) \;&=\; \sum_{a=0}^{n-1}\sum_{b=0}^{n-1}\sum_{c=0}^{n-1} \rho^a \rho^b \rho^c \;=\; \left(1+\rho+\rho^2+\cdots+\rho^{n-1}\right)^3 \\ &=\; \left(1+\rho+\rho^2+\rho^3\right)^3 \\ &=\; -3\sqrt{3}\,i \end{aligned}$$

and

$$B(\rho,\rho,\rho) \;=\; 0 \quad \text{for every choice of the polynomials } P_j.$$

We have annihilated the bricks—so we do not have to worry about where they are. To deal with the unused part, we see that

$$U(\rho,\rho,\rho) \;=\; \sum_{h=1}^{4} \rho^{a_h}\rho^{b_h}\rho^{c_h}$$

is the sum of four powers of ρ and therefore cannot have absolute value greater than 4. However $|C(\rho,\rho,\rho)| = 3\sqrt{3} > 5$. Hence for any choice of (a_h, b_h, c_h), $h = 1, 2, 3, 4$, we have $U(\rho,\rho,\rho) \neq C(\rho,\rho,\rho)$ and, recalling that $B(\rho,\rho,\rho) = 0$,

$$C(\rho,\rho,\rho) \;\neq\; B(\rho,\rho,\rho) + U(\rho,\rho,\rho),$$

contradicting (1). □

Theorem 2 *Let k be a positive integer and let $n = 12k + 3$. You cannot pack $\lfloor n^3/12 \rfloor$ $1 \times 2 \times 6$ bricks into an $n \times n \times n$ cube.*

Proof Assume the cube occupies $[0,n] \times [0,n] \times [0,n]$ and it is packed with $\lfloor n^3/12 \rfloor$ bricks to leave three of its subcubes unoccupied.

We employ the same method as in Theorem 1. With $C(x,y,z)$, $B(x,y,z)$ and $U(x,y,z)$ defined as before,

$$\begin{aligned}
C(\rho,\rho,\rho) &= \sum_{a=0}^{n-1}\sum_{b=0}^{n-1}\sum_{c=0}^{n-1} \rho^a\rho^b\rho^c \\
&= \left(1 + \rho + \rho^2 + \cdots + \rho^{12k+2}\right)^3 \\
&= \left(1 + \rho + \rho^2\right)^3 \;=\; -8, \\
B(\rho,\rho,\rho) &= 0, \\
U(\rho,\rho,\rho) &= \sum_{h=1}^{3} \rho^{a_h}\rho^{b_h}\rho^{c_h},
\end{aligned}$$

where $\rho = 1/2 + \sqrt{3}\,i/2$ and each of a_h, b_h, c_h, $h = 1, 2, 3$, can take any value in $\{0, 1, \ldots, n-1\}$. But for any choice of these parameters,

$$|U(\rho,\rho,\rho)| \;\leq\; 3 \;<\; 8 \;=\; |C(\rho,\rho,\rho)|.$$

Hence

$$C(\rho,\rho,\rho) \;\neq\; B(\rho,\rho,\rho) + U(\rho,\rho,\rho)$$

and therefore the packing does not exist. □

Solution 312.6 – 53 bricks

> You cannot fit 54 $1 \times 1 \times 4$ bricks into a $6 \times 6 \times 6$ box. If you can devise a simple proof, we would like to see it. What about 53 bricks?

Tony Forbes

We show that you cannot pack 53 bricks into a $6 \times 6 \times 6$ cube. There is actually an easy way to prove this by partitioning the cube into 27 coloured $2 \times 2 \times 2$ subcubes. However, in my opinion the somewhat more complicated proof I offer, which is similar to that of Theorem 1 on page 3, is far too interesting to be ignored. We might as well deal with the general case where the cube has side congruent to 2 modulo 4.

Theorem 1 *You cannot pack* $(4m+2)^3/4 - 1$ *bricks of size* $1 \times 1 \times 4$ *into a* $(4m+2) \times (4m+2) \times (4m+2)$ *cube.*

Proof Position the cube to occupy $[0, 4m+2] \times [0, 4m+2] \times [0, 4m+2]$ in Euclidean 3-dimensional space. Suppose $(4m+2)^3/4 - 1$ bricks are packed in the cube to leave 4 units unoccupied.

A point (a, b, c) is represented by the monomial expression $x^a y^b z^c$ in variables x, y, z, The whole cube is represented by the polynomial

$$C(x,y,z) \;=\; \sum_{a=0}^{4m+1} \sum_{b=0}^{4m+1} \sum_{c=0}^{4m+1} x^a y^b z^c.$$

A brick polynomial is one of

$$\begin{aligned} B_1(x,y,z) &= 1 + x + x^2 + x^3, \\ B_2(x,y,z) &= 1 + y + y^2 + y^3, \\ B_3(x,y,z) &= 1 + z + z^2 + z^3, \end{aligned}$$

depending on its orientation. The bricks in the packing are represented by

$$B(x,y,z) \;=\; \sum_{r=1}^{3} P_r(x,y,z) B_r(x,y,z),$$

where $P_1(x,y,z)$, $P_2(x,y,z)$ and $P_3(x,y,z)$ are polynomials. The polynomial representing the holes at (a_h, b_h, c_h), $h = 1, 2, 3, 4$, is

$$U(x,y,z) \;=\; \sum_{h=1}^{4} x^{a_h} y^{b_h} z^{c_h}.$$

For the supposed packing, there must exist polynomials $P_r(x, y, z)$, $r \in \{1, 2, 3\}$, and point coordinates $a_h, b_h, c_h \in \{0, 1, \ldots, 4m+1\}$, $h = 1, 2, 3, 4$, such that

$$C(x, y, z) \;=\; B(x, y, z) + U(x, y, z) \quad \text{for all } x, y, z \in \mathbb{H}, \tag{1}$$

where $\mathbb{H}$ is the ring of quaternions.

We represent a quaternion by an expression of the form $\alpha + \beta i + \gamma j + \delta k$, where α, β, γ, δ are real numbers and i, j and k are the basis elements. Addition is performed by doing each component separately:

$$\begin{aligned} \alpha_1 + &\beta_1 i + \gamma_1 j + \delta_1 k \;+\; \alpha_2 + \beta_2 i + \gamma_2 j + \delta_2 k \\ &= (\alpha_1 + \alpha_2) + (\beta_1 + \beta_2) i + (\gamma_1 + \gamma_2) j + (\delta_1 + \delta_2) k. \end{aligned}$$

Multiplication is done in the usual way,

$$\begin{aligned} (\alpha_1 + \beta_1 i + &\gamma_1 j + \delta_1 k)(\alpha_2 + \beta_2 i + \gamma_2 j + \delta_2 k) \\ &= \alpha_1\alpha_2 + \alpha_1\beta_2 i + \alpha_1\gamma_2 j + \alpha_1\delta_2 k \\ &\quad + \beta_1\alpha_2 i + \beta_1\beta_2 ii + \beta_1\gamma_2 ij + \beta_1\delta_2 ik \\ &\quad + \gamma_1\alpha_2 j + \gamma_1\beta_2 ji + \gamma_1\gamma_2 jj + \gamma_1\delta_2 jk \\ &\quad + \delta_1\alpha_2 k + \delta_1\beta_2 ki + \delta_1\gamma_2 kj + \delta_1\delta_2 kk, \end{aligned}$$

which is then simplified by Table 1. Using Table 1, we see that $B_1(i, j, k) = B_2(i, j, k) = B_3(i, j, k) = 0$ and therefore

$$B(i, j, k) \;=\; 0. \tag{2}$$

Also

$$\begin{aligned} C(i, j, k) &= \sum_{a=0}^{4m+1} \sum_{b=0}^{4m+1} \sum_{c=0}^{4m+1} i^a j^b k^c \\ &= (1+i)(1+j)(1+k) \;=\; (1+i+j+k)(1+k) \\ &= 1 + i + j + k + k - j + i - 1 \;=\; 2i + 2k. \end{aligned} \tag{3}$$

Table 1: Quaternion multiplication

$\times$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

Table 2: Hole coordinates

a mod 2	0	0	0	0	1	1	1	1
b mod 2	0	0	1	1	0	0	1	1
c mod 2	0	1	0	1	0	1	0	1
$i^a j^b k^c$	± 1	$\pm k$	$\pm j$	$\pm i$	$\pm i$	$\pm j$	$\pm k$	± 1

Now consider the holes. We have

$$U(i,j,k) \;=\; \sum_{h=1}^{4} i^{a_h} j^{b_h} k^{c_h}.$$

Apart from a plus or minus sign, $i^a j^b k^c$ depends only on the parities of the coordinates (a,b,c) as indicated in Table 2, which clearly shows that each term of $U(i,j,k)$ must be one of the eight elements of the set

$$R = \{1, -1, i, -i, j, -j, k, -k\}.$$

For (1) to be satisfied, we must have

$$U(i,j,k) \;=\; 2i + 2k. \tag{4}$$

by (2) and (3). The only way to make this quantity from four elements of R, is $i+i+k+k$, and we may assume without loss of generality that

$$i^{a_1} j^{b_1} k^{c_1} \;=\; i^{a_2} j^{b_2} k^{c_2} \;=\; i, \qquad i^{a_3} j^{b_3} k^{c_3} \;=\; i^{a_4} j^{b_4} k^{c_4} \;=\; k.$$

Therefore

$$\begin{aligned} (a_1,b_1,c_1),\ (a_2,b_2,c_2) &\equiv (0,1,1) \text{ or } (1,0,0) \pmod 2, \\ (a_3,b_3,c_3),\ (a_4,b_4,c_4) &\equiv (0,0,1) \text{ or } (1,1,0) \pmod 2. \end{aligned}$$

Now remove the bricks, reflect them in the plane $x = y$ and return them to the cube. The hole that previously occupied position (a_h, b_h, c_h) is now at (b_h, a_h, c_h), $h = 1, 2, 3, 4$. But then

$$\begin{aligned} (b_1,a_1,c_1),\ (b_2,a_2,c_2) &\equiv (0,1,0) \text{ or } (1,0,1) \pmod 2, \\ (b_3,a_3,c_3),\ (b_4,a_4,c_4) &\equiv (0,0,1) \text{ or } (1,1,0) \pmod 2, \end{aligned}$$

and it follows that

$$i^{b_1} j^{a_1} k^{c_1} = \pm j, \quad i^{b_2} j^{a_2} k^{c_2} = \pm j, \quad i^{b_3} j^{a_3} k^{c_3} = \pm k, \quad i^{b_4} j^{a_4} k^{c_1} = \pm k,$$

which contradicts (4). □

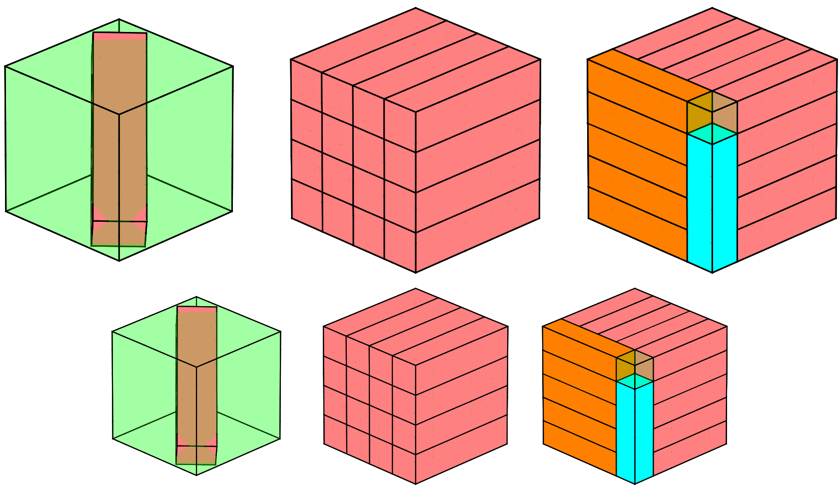
What makes this proof work seems to be the non-symmetry of (3) under permutations of i, j and k. For instance, putting $x = j$ and $y = i$ gives

$$C(j, i, k) \;=\; (1 + j)(1 + i)(1 + k) \;=\; (1 + i + j - k)(1 + k) \;=\; 2 + 2i,$$

and again we can get a contradiction by a suitable transformation of the bricks.

As can be seen from the results summarized in the table below, for the general problem of finding optimal packings of $1 \times 1 \times 4$ bricks in n-sided cubes, the only case where the trivial upper bound is not attained is $n \equiv 2 \pmod 4$.

Cube side, n	Maximum number of bricks	Holes
$n = 0$, 1, 2, 3	0	n^3
$n \equiv 0 \pmod 4$, $n \geq 4$	$n^3/4$	0
$n \equiv 1 \pmod 4$, $n \geq 5$	$\lfloor n^3/4 \rfloor$	1
$n \equiv 2 \pmod 4$	$n^3/4 - 2$	8
$n \equiv 3 \pmod 4$, $n \geq 7$	$\lfloor n^3/4 \rfloor$	3



Problem 315.1 – Rectangles in a square

Tony Forbes

How many $1 \times (n+1)$ rectangles can you fit in an $n \times n$ square?

Obviously fitting any at all might be a bit difficult when $n = 1$ or 2. But as n increases the difference between n and $\sqrt{2}\,n$ becomes more and more significant. There is I think a crossover point at

$$n = \frac{2}{\sqrt{2}-1} = 4.8284,$$

where you can just squeeze in one $(n+1) \times 1$ rectangle along the diagonal.

Problem 315.4 – Cylinder in a cube

What is the largest r such that a cylinder of radius r and length 6 will fit in a cube of side 5?

Obviously this is one of an infinite number of similar problems. I (TF) stumbled upon this particular instance while I was doing something else. It interested me because the answer seems to be 1.0 or thereabouts.