Small regular graphs with girth 5

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We are interested in k-regular graphs with girth 5, large k and small number of vertices. Large means at least 7; small means at most $2k^2$.

A (k, g)-*cage* is a k-regular graph with girth g and the smallest possible number of vertices.

The *Moore bound* for the number of vertices of a k-regular graph with girth g is given by:

$$\begin{aligned} g \text{ odd:} \quad & 1 + k + k(k-1) + k(k-1)^2 + \cdots + k(k-1)^{(g-3)/2}, \\ g \text{ even:} \quad & 2\left(1 + (k-1) + (k-1)^2 + \cdots + (k-1)^{g/2-1}\right). \end{aligned}$$

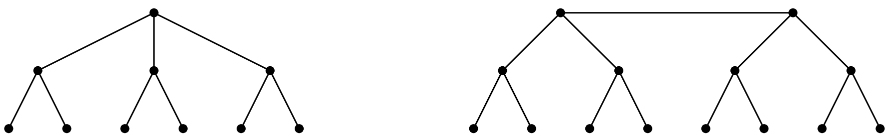

Figure 1

The left-hand diagram in Figure 1 shows how to construct a $(3, 5)$-cage. To complete the graph just add edges to the bottom row of vertices to make the thing 3-regular whilst avoiding triangles and squares. The result will be the Petersen graph.

The right-hand diagram is for a (3,6)-cage.

When $g < 5$ the constructions are even simpler.

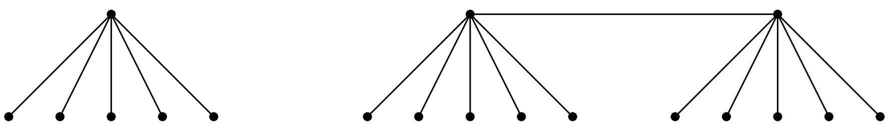

Figure 2

A $(k, 3)$-cage is a complete graph K_{k+1}. In the left-hand diagram of Figure 2 you can join each vertex in the bottom row to every other vertex in the bottom row.

A $(k, 4)$-cage is a complete bipartite graph $K_{k,k}$. In the right-hand diagram of Figure 2 you can join each vertex on the left in the bottom row to every vertex in the bottom row on the right without introducing a triangle.

Thus we have two infinite sets of cages. For another, we skip 5 and go to $g = 6$.

Take a projective plane of order $k - 1$, which has $k^2 - k + 1$ points and the same number of lines. Moreover, every line has k points and every point is incident with k lines. Construct the *incidence graph* of the projective plane. The vertices are the points and lines, and there is an edge $i \sim j$ whenever line i is incident with point j.

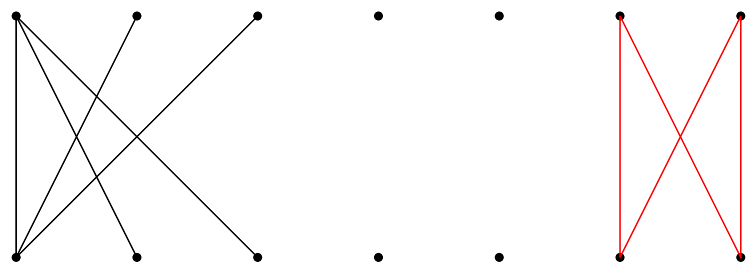

Figure 3

The graph is bipartite and is k-regular. The bipartiteness ensures that there are no triangles and pentagons. There are no squares either. A 4-cycle such as the red one in Figure 3 would imply the existence of two lines each containing the same pair of points. This cannot occur in a projective plane.

The graph can't have girth greater than 6 because there are not enough vertices. Hence it has girth 6, and since the number of vertices, $2(k^2-k+1) = 2(1+(k-1)+(k-1)^2)$, attains the Moore bound it is a $(k, 6)$-cage.

If $g = 6$ and $k = 13$, the previous construction won't work because there is no known projective plane of order 12. But we can get quite a small graph by using instead a 13-GDD of type 13^{13}, also known as a transversal design, TD(13,13). The design has $13^2 = 169$ points and 169 blocks[1]. Each block has 13 points and each point meets 13 blocks. Hence its incidence graph is 13-regular, is bipartite and, by the same argument as for projective planes, has girth 6. It has 338 points; compare with the Moore bound, 314.

More generally, we can perform this construction whenever k is a prime power.

$g = 5$

The Moore bound for $g = 5$ is $1 + k + k(k-1) = k^2 + 1$. However, this is attained only for:

$k = 2$, the pentagon,

$k = 3$, the Petersen graph,

$k = 7$, the Hoffman–Singleton graph,

and possibly $k = 57$.

A k-regular Moore graph with girth 5 has diameter 2, as is clear from the left-hand diagram in Figure 1, and its adjacency matrix, A, satisfies

$$A^2 = kI + J - I - A,$$

where I is the identity matrix and J is the all-ones matrix. To see this, recall that $[A^2]_{i,j}$ is the number of 2-step walks from vertex i to vertex j. So:

$[A^2]_{i,i} = k$ since the only way to get from i to i in 2 steps is to go to a neighbour of i and return;

$[A^2]_{i \neq j, i \sim j} = 0$ since a 2-edge path from i to j via a common neighbour would create a triangle;

[1]The words *line* and *block* mean the same thing

$[A^2]_{i\neq j,\ i\not\sim j} = 1$ since (i) because the graph has diameter 2 there must be a 2-edge path i to j via a common neighbour, and (ii) a second such path would create a 4-cycle.

Let $\mathbf{a}$ be an eigenvector of A that is orthogonal to $\mathbf{j}$, the all-ones vector. Let α be an eigenvalue of $\mathbf{a}$. Then

$$A^2\mathbf{a} = k\mathbf{a} + 0 - \mathbf{a} - A\mathbf{a},$$
$$\alpha^2 = k - 1 - \alpha.$$

Then after some elementary but messy number theory, one is led to the stated four possible values of k, namely 2, 3, 7 and 57. See Peter Cameron's *Combinatorics*, §11.12.

For an interesting (to me) characterization of regular graphs with girth at least 5, we have the following.

Let A be the adjacency matrix of a graph G with v vertices. Then G is k-regular and has girth at least 5 iff every row of $A^2 - A$ has the frequencies

$$(-1)^k \quad 0^{v-k^2-1} \quad 1^{k(k-1)} \quad k^1; \tag{1}$$

i.e. k occurrences of -1, $v - k^2 - 1$ occurrences of 0, $k(k-1)$ occurrences of 1 and one occurrence of k.

To see this, consider row i of the matrix $A^2 - A$ of a k-regular graph with girth at least 5. Again recall that $[A^2]_{i,j}$ is the number of 2-step walks from i to j. Then:

$[A^2 - A]_{i,i} = k$ since $[A]_{i,i} = 0$ and $[A^2]_{i,i} = k$ as before; thus the k^1 in the frequency specification corresponds to the k in column i;

there are k vertices adjacent to i and for these we have $[A^2 - A]_{i\neq j,\ i\sim j} = -1$ since $[A]_{i\sim j} = 1$ and there is no way to get from i to j in 2 steps without creating a triangle;

from i there are $k(k-1)$ vertices $j \neq i$ that you can get to in 2 steps, and for these we have $[A^2 - A]_{i\neq j,\ i\not\sim j} = 1$;

if there are any neighbours of i left over, they represent $i \neq j$, $i \not\sim j$, j is at distance $>$ 2 from i, and for these we have $[A^2 - A]_{i,j} = 0$.

For the converse, assume that the rows of $A^2 - A$ have the frequency specification (1).

Consider row i. The $(-1)^k$ in (1) must come from k vertices j, where $j \sim i$ and j and i have no common neighbour. Therefore $\deg(i) \geq k$. Hence from k^1 in (1) we have $\deg(i) = k$. Since every $j \sim i$ has no neighbour in common with i, there cannot be any triangles (i, j, s). Hence the graph is k-regular and triangle-free. Now suppose there exists a 4-cycle (i, s, j, t), say. Then, since there are no triangles, $i \not\sim j$. However, there are at least two 2-paths from i to j, and hence $[A^2 - A]_{i,j} \geq 2$, contradicting (1).

Observe that the number of zeros in row i cannot be negative. Therefore $v \geq k^2 + 1$, confirming that v cannot be less than the Moore bound.