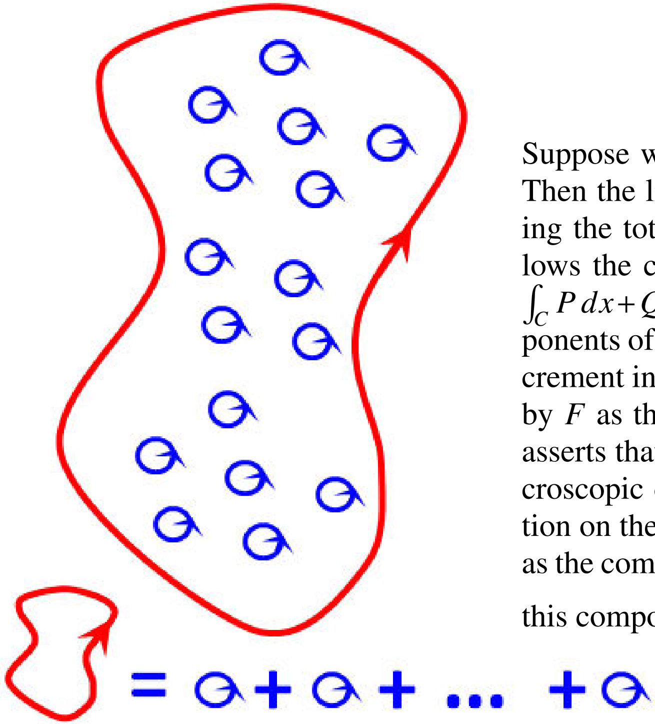
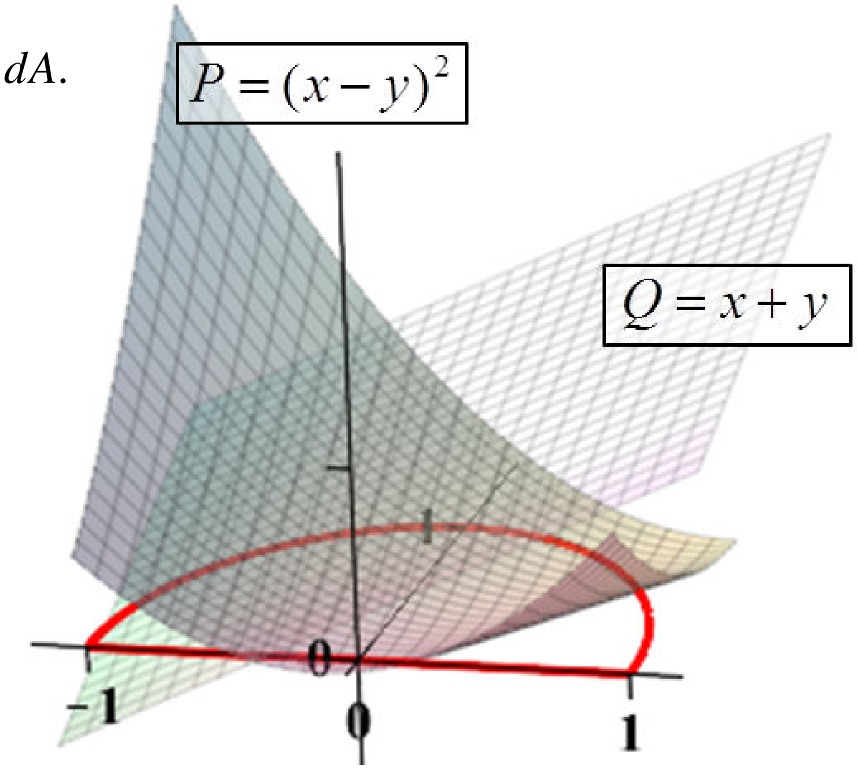
THEOREM OF THE DAY

Green's Theorem *Let C be a closed, anticlockwise-oriented curve in the xy-plane enclosing a region D. Let $F(x,y) = (P(x,y), Q(x,y))$ be a 2-valued function having continuous partial derivatives on C and inside D. Then*

$$\int_C F d\mathbf{s} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$



Suppose we think of F as a force field acting in the plane. Then the line integral $\int_C F d\mathbf{s}$ may be thought of as measuring the total work done by F acting on a particle as it follows the curve C; the integral is often written in the form $\int_C P\,dx + Q\,dy$, making explicit the action of the x and y components of the force as the particle moves through a small increment in the x and y directions. We refer to this work done by F as the 'circulation of F around C'. Green's Theorem asserts that circulation around C is the accumulation of 'microscopic circulations' around points in D: see the illustration on the left; these microscopic circulations are measured as the component perpendicular to the plane of the **curl** of F; this component is calculated as: $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$.



As an illustration we take $P(x,y) = (x-y)^2$ and $Q(x,y) = x + y$. These are plotted as surfaces in 3D on the right, and a half-unit circle, closed by adjoining a segment of the x-axis, is chosen as the curve C. We find $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1 + 2(x-y)$ and Green's Theorem yields the value of $\int_C F d\mathbf{s}$ by double integration as

$$\iint_D 1 + 2(x-y)\,dA = \int_{-1}^{1}\int_0^{\sqrt{1-x^2}} 1 + 2(x-y)\,dy\,dx = \int_{-1}^{1}\left(\sqrt{1-x^2} + 2x\sqrt{1-x^2} - 1 + x^2\right) dx = \left[\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x - \frac{2}{3}(1-x^2)^{3/2} - x + \frac{x^3}{3}\right]_{-1}^{1} = \frac{\tau}{4} - \frac{4}{3}.$$

Evaluating the line integral directly requires piecewise parameterisation of the curve C: half-circle $c_1(t) = (-\cos t, \sin t), 0 \le t \le \tau/2$ and base $c_2(t) = (t, 0), -1 \le t \le 1$:

$$\begin{aligned}\int_C F d\mathbf{s} &= \int_0^{\tau/2} F(c_1(t)) \cdot c_1'(t)\,dt + \int_{-1}^{1} F(c_2(t)) \cdot c_2'(t)\,dt = \int_0^{\tau/2} F(\cos t, \sin t) \cdot (-\sin t, \cos t)\,dt + \int_{-1}^{1} F(t,0) \cdot (1,0)\,dt \\ &= \int_0^{\tau/2} (\cos t - \sin t)^2(-\sin t) + (\cos t + \sin t)\cos t\,dt + \int_{-1}^{1} t^2\,dt = -2 + \frac{\tau}{4} + \frac{2}{3} = \frac{\tau}{4} - \frac{4}{3}, \quad \text{as expected.}\end{aligned}$$

George Green published this theorem, a powerful generalisation of the Fundamental Theorem of the Calculus, in 1828.

Web link: mathinsight.org/greens_theorem_idea (on which the above description and examples are based).

Further reading: *Inside Interesting Integrals* by Paul J. Nahin, Springer, 2015, Chapter 8.