M500 Problem: Colouring pairs

In chapter 7 of Joseph G. Rosenstein's book *Linear Orderings*, Academic Press, 1982, he poses a nice problem which may be stated thus: suppose we list the unordered pairs of natural numbers in the order $\{i, m-i\}, 0 \le i \le \lfloor (m-1)/2 \rfloor, m = 0, \ldots, \infty$. Colour all entries at odd positions in this listing Red and colour Blue the entries in even positions. So the first six Red pairs are $\{0,1\}, \{0,3\}, \{0,4\}, \{0,5\}, \{2,3\}, \{1,5\}$, and these alternate with the first six Blue pairs $\{0,2\}$, $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{0,6\}$, $\{2,4\}$. Now Rosenstein's problem is to specify an infinite subset of the natural numbers for which every unordered pair is the same colour. In fact, it isn't very difficult to solve this for the colour Red. But it is not just difficult but impossible to solve for the colour Blue, and this suggests your M500 problem: show that if X is a set of natural numbers all of whose unordered pairs are Blue then X has at most four elements.

Putative answer (to check I'm right and in case nothing better materialises)

Consider a set X for which ever pair $\{a, b\}, a < b$, is Blue, that is, in Rosenstein's listing its position, call it $p(a, b)$, is even. Unless X has only two elements there will be some pair for which $a + b$ is even. Then the position of this pair in Rosenstein's listing may be calculated to be

$$p(a,b) = \left(\frac{a+b}{2}\right)^2 + a + 1. \tag{1}$$

There are two possibilities:

a **odd:** then b is odd and for $p(a, b)$ to be even, by (1), $a + b$ must be a multiple of 4, so

$$a + b = 2m + 1 + 2n + 1 = 4K, \text{ so } m + n \text{ is odd, say } a = 2(2p + 1) + 1, b = 2(2q) + 1.$$

Then

$$X \supseteq \{4p + 3, 4q + 1\}. \tag{2}$$

a **even:** then b is even and by (1), $a + b$ cannot be a multiple of 4, so

$$a + b = 2m + 2n = 4K + 2 \text{ so } m + n \text{ is odd, say } a = 2(2r), b = 2(2s + 1).$$

Then

$$X \supseteq \{4r, 4s + 2\}. \tag{3}$$

Now in the case where a, b are odd, we find that our set X can have no further odd elements without creating a pair contradicting (2). For even a, b we can have no further even elements without contradicting (3). So we can have at most four elements in total, X having the form:

$$X = \{4p + 3, 4q + 1, 4r, 4s + 2\}.$$