



THEOREM OF THE DAY



Goodstein's Theorem For a positive integer M , derive the hereditary base k representation, in which every numeral is either k or zero, as follows: (1) write M in base k , as $M = a_0k^0 + a_1k^1 + \dots + a_{n-1}k^{n-1} + a_nk^n$, for suitable a_i in $\{0, \dots, k-1\}$; (2) for $0 \leq i \leq n$, write a_ik^i as a sum of a_i copies of k^i ; (3) apply steps (1) – (3) to all occurrences of $1, \dots, k-1$ in exponents. Now suppose the Goodstein sequence of M is obtained by applying the following algorithm, starting at base $k = 2$:

- (G1) write M in hereditary base k notation;
 (G2) replace every occurrence of ' k ' by ' $k+1$ ' and subtract 1 from the resulting number;
 (G3) if the result is zero then STOP; otherwise apply (G1) – (G3) to the result with base k replaced by base $k+1$.

Then, for every positive integer M , the Goodstein sequence terminates.

Hereditary base 5 example: $15678 \xrightarrow{(1)} 3 \times 5^0 + 2 \times 5^2 + 1 \times 5^6 \xrightarrow{(2)} 5^0 + 5^0 + 5^0 + 5^2 + 5^2 + 5^6 \xrightarrow{(3)} 5^0 + 5^0 + 5^0 + 5^{5^0+5^0} + 5^{5^0+5^0} + 5^{5^0+5^1} \xrightarrow{(3)} 5^0 + 5^0 + 5^0 + 5^{5^0+5^0} + 5^{5^0+5^0} + 5^{5^0+5^0}$

		3	3	3	2	1
	k	2	3	4	5	6
hereditary base k		$2^{2^0} + 2^0$	3^{3^0}	$4^0 + 4^0 + 4^0$	$5^0 + 5^0$	6^0
$k \rightarrow k+1$		$3^{3^0} + 3^0$	4^4^0	$5^0 + 5^0 + 5^0$	$6^0 + 6^0$	7^0
subtract 1		3^{3^0}	$4^{4^0} - 1$	$5^0 + 5^0$	6^0	0
		3	3	2	1	0

		6	29	257	3125	46
	k	2	3	4	5	6
hereditary base k		$2^{2^{2^0}} + 2^{2^0}$	$3^{3^{3^0}} + 3^{3^0} + 3^0$	$4^{4^{4^0}} + 4^0$	$5^{5^{5^0}}$...
$k \rightarrow k+1$		$3^{3^{3^0}} + 3^{3^0}$	$4^{4^{4^0}} + 4^0 + 4^0$	$5^{5^{5^0}} + 5^0$	$6^{6^{6^0}}$...
subtract 1		$3^{3^{3^0}} + 3^{3^0} - 1$	$4^{4^{4^0}} + 4^0$	$5^{5^{5^0}}$	$6^{6^{6^0}} - 1$...
		29	257	3125	46655	...

The Goodstein sequence for $M = 3$ is seen here to terminate after 5 iterations of the algorithm but this is the largest value of M for which the Goodstein sequence can feasibly be constructed: even for $M = 4$ it requires many more iterations than there are atoms in the universe!

Goodstein's 1944 theorem is important because it can be stated as a sentence in Peano arithmetic but no proof exists within this system. It is therefore a 'natural' example of Gödel's First Incompleteness Theorem in action.

Web link: old.nationalcurvebank.org/goodstein/goodstein.htm

Further reading: *An Introduction to Formal Logic* by Peter Smith, Cambridge University Press, 2003.

