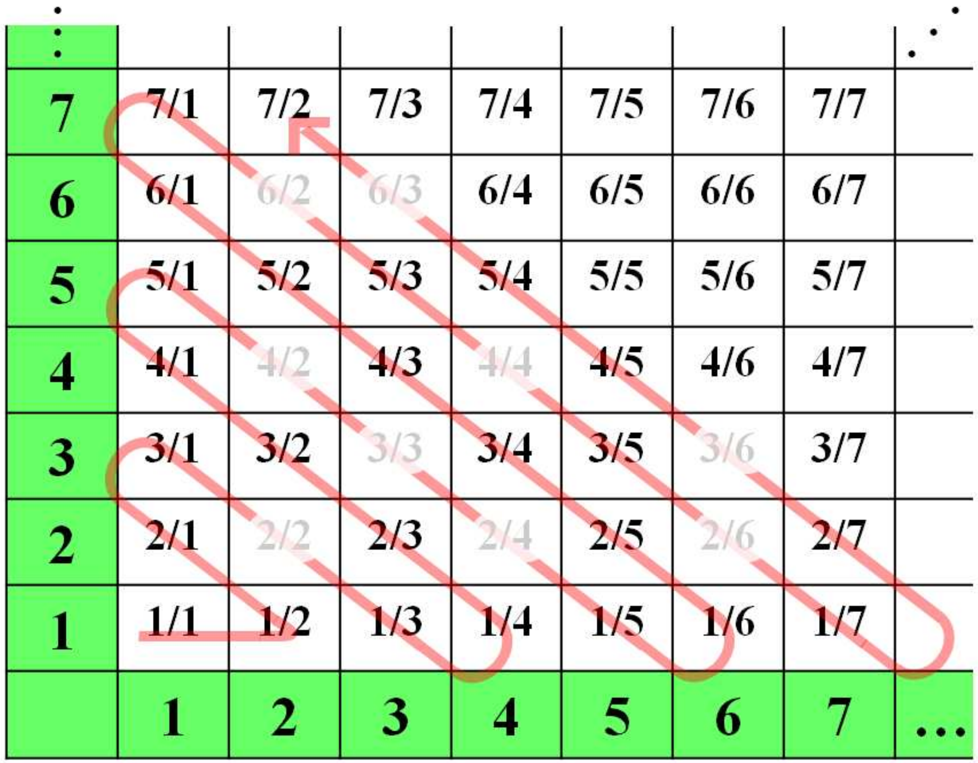
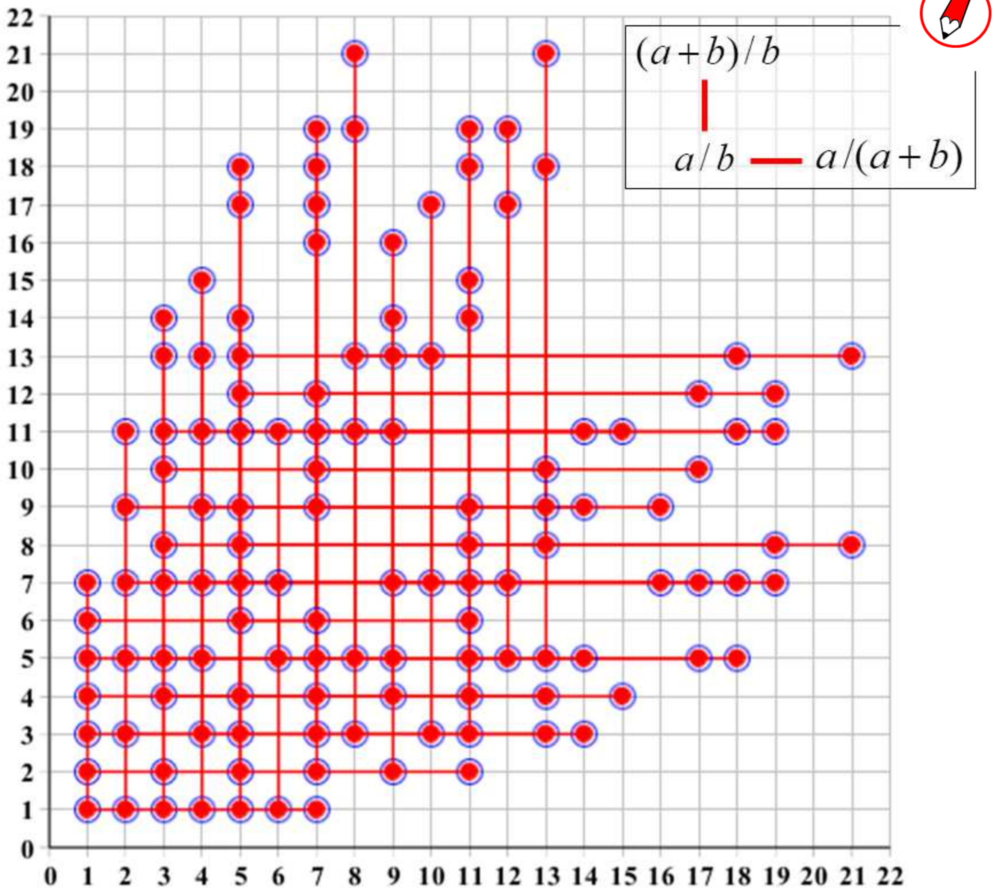
THEOREM OF THE DAY

Countability of the Rationals *There is a one-to-one correspondence between the set of positive integers and the set of positive rational numbers.*





The rational numbers are those real numbers which may be written as a quotient of two integers. This suggests a way, shown above, to display the positive rationals: in a grid whose rows and columns are indexed by the positive integers. Such a display would seem to confirm that the set of positive rationals is a whole 'dimension' bigger than the set of positive integers. Nevertheless, the sets are the same size in the sense that for every positive integer there is a positive rational and vice versa, in other words, there is a one-to-one correspondence between the two sets. This is proved by counting through the grid along a zigzag path, as depicted above. Corresponding to the integer 14, for example, we have the quotient 3/4. Notice that our count must 'skip over' fractions not in their lowest form, to avoid repeats (e.g. 2/4 = 1/2).

An enumeration that skips over unwanted objects is fine. For instance we can certainly count prime numbers by skipping all composite integers, even though we may have no rule that says when the next skip occurs. In fact, for counting the rationals there *is* a rule: a clever construction that avoids non-lowest-form fractions. Branching out from 1/1 we build a tree using the rule given in the top right corner. The tree will include every fraction exactly once in its lowest form! Our count of the rationals is now a level-by-level traversal of the tree. The n-th level (with 1/1 at level 0) adds 2^n new fractions to the count. The tree, up to level 6, is plotted above, superimposed on the same integer lattice as we used above left.

Cantor asserted this result in correspondence with Richard Dedekind in the early 1870s. The zigzag countability proof on the left extends to countable unions of countable sets, although the axiom of choice must be invoked. The tree of fractions goes back to Moritz Abraham Stern (1858) and Achille Brocot (1860).

Web link: mathlesstraveled.com/2007/12/27.
Further reading: *Sets, Logic and Categories* by Peter Cameron, Springer, 1999.