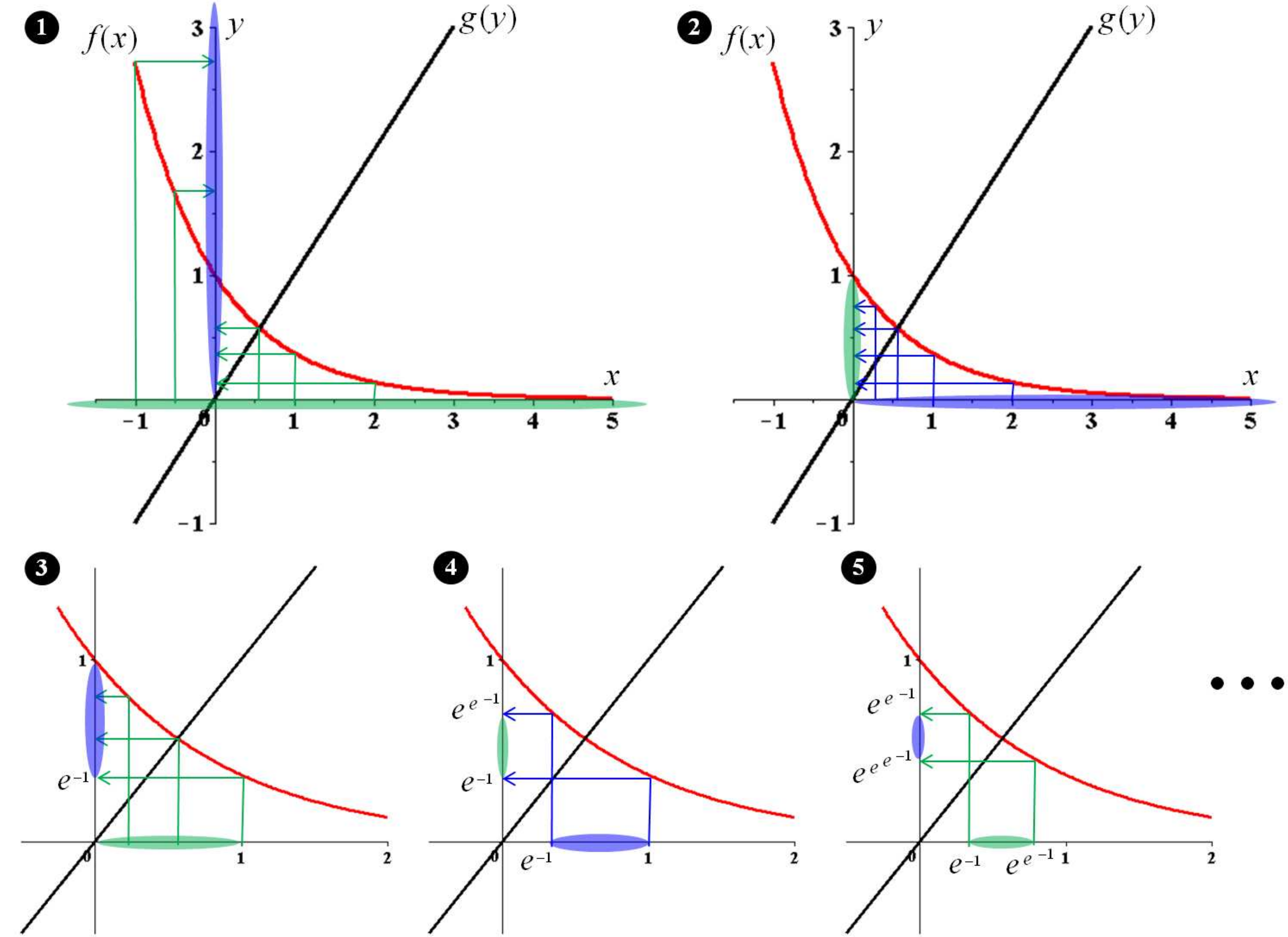
THEOREM OF THE DAY

The Cantor–Bernstein–Schröder Theorem *Let A and B be sets for which there exist injective mappings from A to B and from B to A. Then there is a bijective correspondence between A and B.*



We have chosen here a very simple example but one which allows us to follow through the proof of the theorem. Our sets A and B are the real numbers $\mathbb{R}$, with A represented by the horizontal, x, axis and B by the vertical, y, axis. Our injections are $f : A \to B$ and $g : B \to A$ defined by $f(x) = e^{-x}$ and $g(y) = y$, respectively. Of course, g is already a bijection between A and B: it matches every point on the y axis with the exactly corresponding point on the x axis. But f is not: it maps the x axis to the positive y axis, as indicated by the green arrows at ❶. The proof extends f to a bijection by combining it with g.

The idea is to define a function F on subsets of A thus: $F(X) = A \backslash g(B \backslash f(X))$. At ❶ and ❷ this is shown for $X = A$, the whole x axis: $f(X)$ is the positive y axis; $B \backslash f(X)$ is the non-positive y axis; then $g(B \backslash f(X))$ is the non-positive x axis; finally $F(X)$ is the positive x axis.

From ❷ onwards this is iterated: by ❺ we have constructed $F^4(A)$ and are in the process of constructing $F^5(A)$. The crux is, it can be shown that $A_0 = A \cap F(A) \cap F^2(A) \cap \ldots$ is a fixed point of F, i.e. $F(A_0) = A_0$. This means that $A \backslash g(B \backslash f(A_0)) = A_0$ so $g(B \backslash f(A_0)) = A \backslash A_0$. Et voilà,

$$f_0(x) = \begin{cases} f(x) & \text{if } x \in A_0 \\ g^{-1}(x) & \text{if } x \in A \backslash A_0 \end{cases}$$

is a well-defined bijection from A to B. In our example A_0 is a single point: the solution to the equation $e^{-x} = x$, the so-called *Omega constant*, $\Omega \approx 0.57$. Result: in our example, f_0 and g are identical!

We write $A < B$ if A maps injectively to B. The theorem justifies the notation: $A < B$ and $B < A$ if and only if $A = B$. This reflects a far-reaching step in Cantor's invention of cardinal numbers.

Web link: www.cut-the-knot.org/WhatIs/Infinity/CBS.shtml. The proof above is probably due to Richard Dedekind in 1899, see rjlipton.wordpress.com/2014/07/31.

Further reading: *Proofs of the Cantor-Bernstein Theorem: A Mathematical Excursion* by Arie Hinkis, Birkhäuser; 2013.