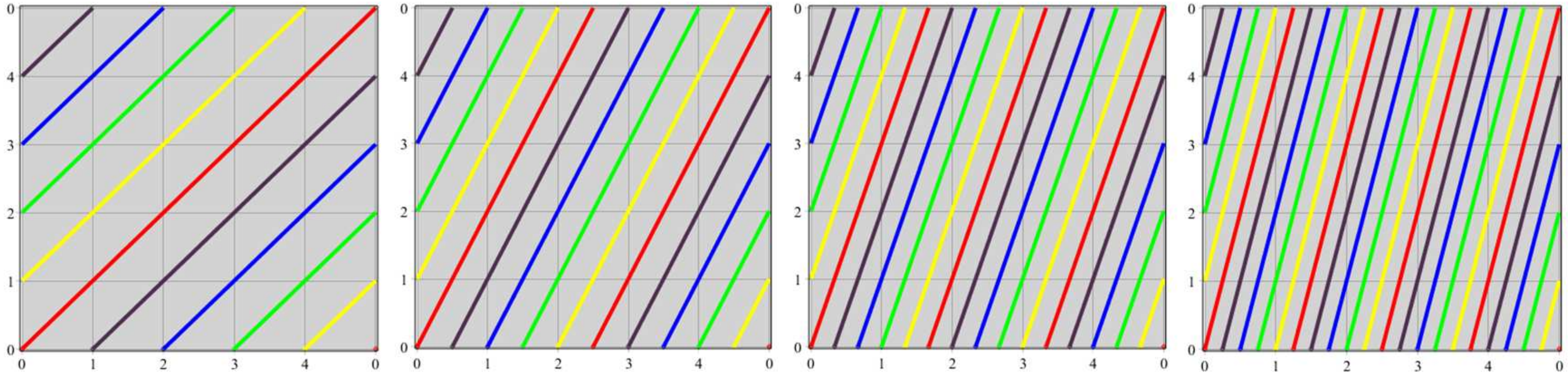
THEOREM OF THE DAY

The Third Isomorphism Theorem *Suppose that K and N are normal subgroups of group G and that K is a subgroup of N. Then K is normal in N, and there is an isomorphism from $(G/K)/(N/K)$ to G/N defined by $gK \cdot (N/K) \mapsto gN$.*



$x + b \bmod 5,\ b = 0,..,4$ $2x + b \bmod 5,\ b = 0,..,4$ $3x + b \bmod 5,\ b = 0,..,4$ $4x + b \bmod 5,\ b = 0,..,4$

To illustrate we take G to be the Frobenius group F_{20}, which may be defined as the group of affine linear maps $x \mapsto ax + b \mod 5$, $a = 1, \ldots, 4$, $b = 0, \ldots, 4$. In this group we have:

Multiplication: by function composition, applied on the right, thus: $(ax + b) \circ (cx + d) = c(ax + b) + d = cax + cb + d$, with all arithmetic modulo 5.
Inverses: calculated as $(ax + b)^{-1} = a^{-1}x - a^{-1}b$, where a^{-1} multiplies with a to give $1 \mod 5$. E.g., $3^{-1} \mod 5 = 2$ so $(3x + 4)^{-1} = 2x - 8 \mod 5 = 2x + 2$.
Conjugation: calculated as $(ax + b)^{-1} \circ (cx + d) \circ (ax + b) = cx - cb + ad + b \mod 5$. For example, $(3x + 4)^{-1} \circ (4x + 2) \circ (3x + 4) = 4x - 16 + 6 + 4 \mod 5 = 4x + 4$.
Notice that conjugation respects the plots above — it takes an affine map to another in the same plot. Inversion respects the first and fourth plots but exchanges between plots two and three.

Now take the set N to be the maps in the first and fourth plots above. Thus $N = \{ax + b : a = 1, 4; b = 0, 1, 2, 3, 4\}$. It is closed under composition and inverses (so is a subgroup) and conjugation over G (so is a normal subgroup). The same is true of the subset K of N consisting of just the first plot above, defined by $K = \{x + b : b = 0, 1, 2, 3, 4\}$. So the isomorphism theorem applies and $(G/K)/(N/K) \cong G/N$.

The elements of G/K have the form $(ax + b) \circ K$, a 1–1 mapping of the first plot to the a-th plot. So $G/K = \{\ , 2\ , 3\ , 4\ \}$, with mod 5 multiplication, giving the cyclic group of order 4. The other quotient on the left of the isomorphism, N/K is, similarly, the cyclic group of order 2: $N/K = \{\ , 4\ \}$. The quotient of G/K by N/K is constructed following the pattern of this example: $2\ \times \{\ , 4\ \} = \{2\ , 3\ \}$. We cycle between two different cosets, giving the cyclic group of order 2. And here, finally, is the right-hand side, $G/N = \{\{\ ,\ \}, 2\{\ ,\ \}\}$; the cosets are different but the 2-cycle is the same!

This theorem, as with the second isommorphism theorem, is (a) due in its most general form to Emmy Noether in 1927 and (b) a corollary of the first isomorphism theorem.

Web link: people.reed.edu/~jerry/332/09isom.pdf
Further reading: *Introduction to Algebra, 2nd Edition* by Peter J. Cameron, OUP, 2007.