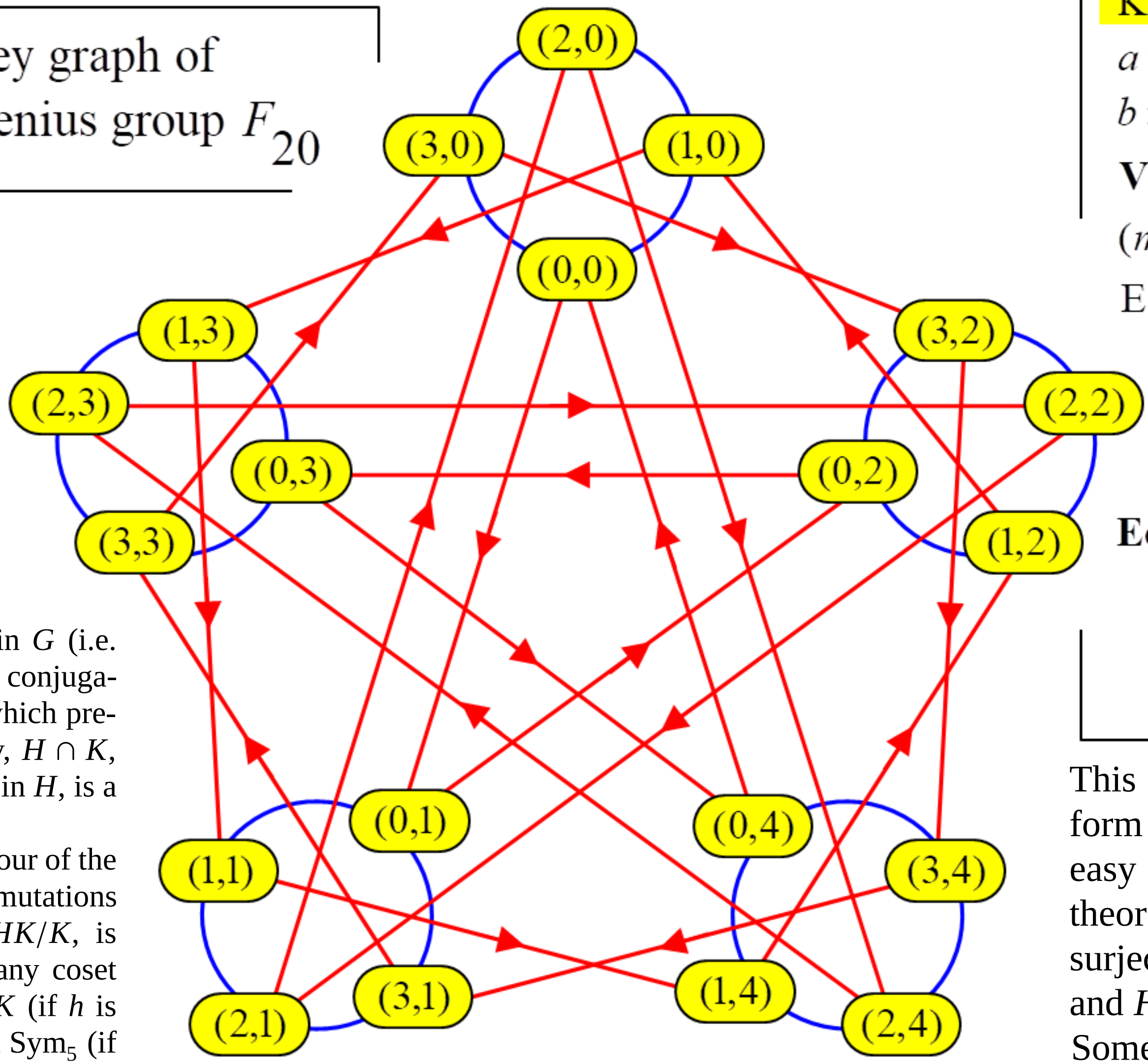
THEOREM OF THE DAY

The Second Isomorphism Theorem *Suppose H is a subgroup of group G and K is a normal subgroup of G. Then HK is a group having K as a normal subgroup, $H \cap K$ is a normal subgroup of H, and there is an isomorphism from $H/(H \cap K)$ to HK/K defined by $h(H \cap K) \mapsto hK$.*

Cayley graph of Frobenius group F_{20}



KEY

$a := (1\,2\,3\,4\,5)$

$b := (1\,2\,5\,4)$

Vertices (group elements)

$(m,n) := b^m a^n = (1\,2\,5\,4)^m (1\,2\,3\,4\,5)^n$

E.g. $(1,2) = b^1 a^2$

$= (1\,2\,5\,4)(1\,2\,3\,4\,5)^2$

$= (1\,2\,5\,4)(1\,3\,5\,2\,4)$

$= (1\,4\,3\,5)$

Edges (left multiplication)

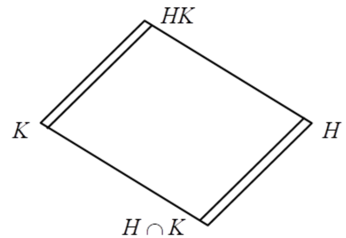
→ by a

— by b (anticlockwise cycle)

To illustrate we take G to be Sym_5, the group of 5! permutations of $\{1, 2, 3, 4, 5\}$. The Frobenius group F_{20} may be defined as a subgroup H of G generated by a 5-cycle, a, and a 4-cycle, b, satisfying $(ab)^4 = a(ab)(ba)^{-1} = 1$. We take K to be Alt_5, the subset of 5!/2 even permutations: identity, 5-cycles, and products of two 2-cycles. This is normal in G (i.e. $g^{-1}Kg = K$ for all g) because conjugation, $g^{-1}xg$, is a 1–1 mapping which preserves cycle structure; similarly, $H \cap K$, the subset of even permutations in H, is a normal subgroup of H.

Now we can uncover the behaviour of the normal subgroup of even permutations of F_{20}. The target quotient, HK/K, is $\mathrm{Sym}_5/\mathrm{Alt}_5 \cong C_2$ because (1) any coset of K, say, hK, is either all of K (if h is even) or all odd permutations in Sym_5 (if h is odd), and (2) multiplication of cosets mirrors addition modulo 2: $hK.h'K = hh'K$ switches coset if and only if h and h' have different parity. So cosets of $H \cap K$ must behave in exactly the same way in H. And we can see this in the Cayley graph: $H \cap K$ is the 'identity' coset consisting of all vertices (m, n), m even; there is one other coset, $b(H \cap K)$, and left multiplication by b cycles between the two.

This theorem, due in its most general form to Emmy Noether in 1927, is an easy corollary of the first isomorphism theorem. Thus, if $f : H \to HK/K$ is the surjective homomorphism $h \mapsto hK$ then and $H/\ker f \cong \mathrm{im}\, f$ and $\ker f = H \cap K$. Some refer to it as a 'parallelogram rule', associating it with the diagram on the right.



Web link: people.reed.edu/~jerry/332/09isom.pdf

Further reading: *Classic Algebra* by P.M. Cohn, John Wiley & Sons, 2000, Chapter 9.