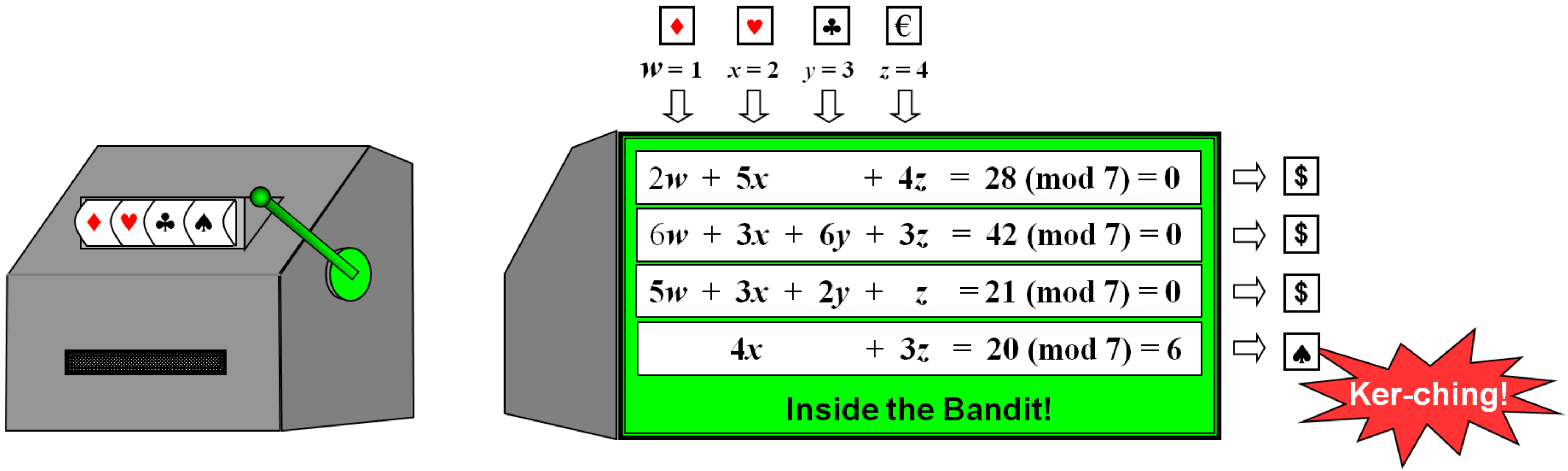
THEOREM OF THE DAY

A Theorem of Melody Chan on Group Actions *Let n be a positive integer, and F a field, either infinite or finite with $|F| > n+1$. Then the members of the set F^n of n-vectors can each be coloured Green or Blue in such a way that, for any non-singular $n \times n$ matrix B with entries in F, other than the identity matrix, there is some Green vector v for which Bv is Blue.*



Melody Chan's theorem is a recipe for making a one-arm bandit! In the above example we make a bandit which chooses $n = 4$ symbols from the set {♦, ♥, ♣, ♠, \$, €, £} (7 being the cardinality of the smallest finite field greater than $4+1$). As the four symbols spin past you pull the handle to choose a 4-vector. Inside the machine the symbols are translated into numbers (I used \$ → 0, ♦ → 1, ♥ → 2, ♣ → 3, € → 4, £ → 5 and ♠ → 6) and fed into a set of four simultaneous equations (the matrix B). The symbols spin on and eventually stop at the values given by the right hand side of the equations. Because our field is GF(7), all the calculations are done mod 7. Now a set of 'Blue' winning vectors is guaranteed by the theorem: even if the equations in the machine are constantly reset (provided they continue to give a non-singular, non-identity B) a certain number of (Green) winning inputs is guaranteed to map to these (Blue) winning outputs.

In fact, Chan's proof of the theorem is constructive, specifying a set of Blue vectors. For our bandit, regardless of its equations, they are always (♦, \$, \$, \$), (\$, ♦, \$, \$), (\$, \$, ♦, \$), (\$, \$, \$, ♦) (these could pay \$1, say); (\$, ♥, \$, \$), (\$, \$, ♥, \$), (\$, \$, \$, ♥) (say, \$2); (\$, \$, ♣, \$), (\$, \$, \$, ♣) (\$10); and (\$, \$, \$, ♠) (JACKPOT!). There are ten winning inputs out of the 2401 possible choices of 4-vector.

Melody Chan proved this theorem in 2004 while still an undergraduate student at Yale University.

Web link: see paper R70 at www.combinatorics.org/Volume_13/v13i1toc.html.

Further reading: *Fearless Symmetry: Exposing the Hidden Patterns of Numbers* by Avner Ash and Robert Gross, Princeton University Press, 2006.