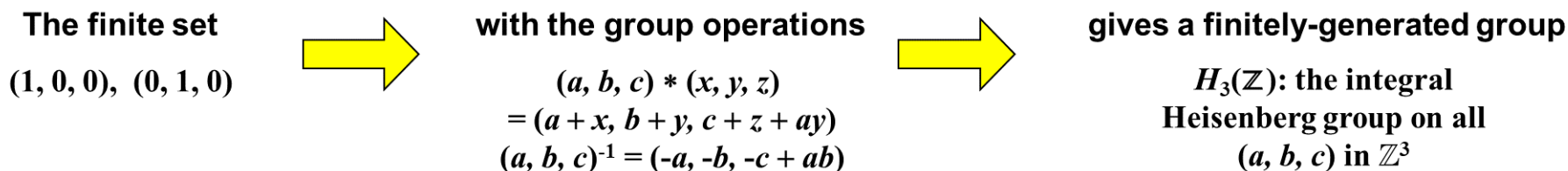


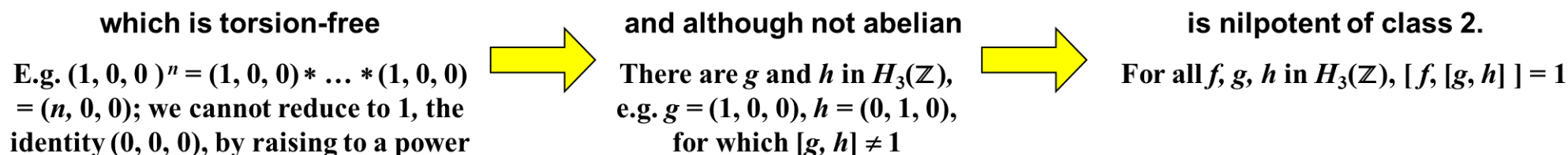


THEOREM OF THE DAY

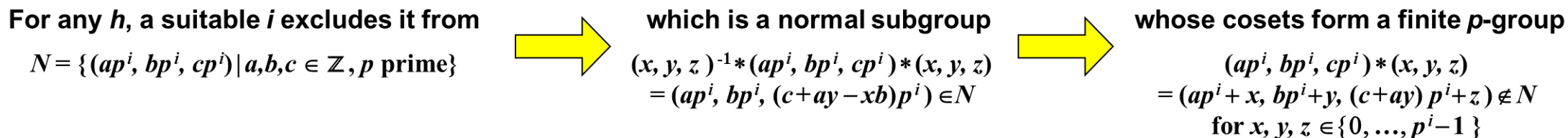
Gruenberg's Theorem on Nilpotent Groups A finitely generated, torsion-free, nilpotent group is a residually finite p -group, for every prime p .



E.g. $(0, 0, 1) = (1, 0, 0) * (0, 1, 0) * (1, 0, 0)^{-1} * (0, 1, 0)^{-1} = [(1, 0, 0), (0, 1, 0)]$, with $[g, h] = ghg^{-1}h^{-1}$, the commutator of g and h .



E.g. $(0, 0, 1) = [(1, 0, 0), (0, 1, 0)]$ commutes with any f in $H_3(\mathbb{Z})$, $[f, (0, 0, 1)] = 1$. The set $\{(0, 0, a) \mid a \in \mathbb{Z}\}$ is the centre of $H_3(\mathbb{Z})$.



Which means, precisely, that we have a residually finite p -group

The example above is by no means atypical — in fact, every finitely generated, torsion-free, nilpotent group is isomorphic to a subgroup of a Heisenberg group over the integers. The theorem essentially says that every element in such a group ‘lives inside’ a finite p -group. Equivalently, any such group can be embedded into a direct product of finite p -groups.

With this 1957 theorem, Karl Gruenberg showed that the study of an important class of nilpotent groups could be essentially reduced to the study of finite p -groups, whose properties had been the subject of deep investigation since Cauchy’s work in the 1840s.

Web link: projecteuclid.org/euclid.bams/1183530287

Further reading: *A Course in the Theory of Groups, 2nd ed.* by DJS Robinson, Springer-Verlag, 1998, chapter 5.

