THEOREM OF THE DAY

Gruenberg's Theorem on Nilpotent Groups *A finitely generated, torsion-free, nilpotent group is a residually finite p-group, for every prime p.*

The finite set

$(1, 0, 0), \ (0, 1, 0)$

with the group operations

$(a, b, c) * (x, y, z)$
$= (a + x, b + y, c + z + ay)$
$(a, b, c)^{-1} = (-a, -b, -c + ab)$

gives a finitely-generated group

$H_3(\mathbb{Z})$: the integral Heisenberg group on all (a, b, c) in $\mathbb{Z}^3$

E.g. $(0, 0, 1) = (1, 0, 0) * (0, 1, 0) * (1, 0, 0)^{-1} * (0, 1, 0)^{-1} = [(1, 0, 0), (0, 1, 0)]$, with $[g, h] = ghg^{-1}h^{-1}$, the *commutator* of g and h.

which is torsion-free

E.g. $(1, 0, 0)^n = (1, 0, 0) * \ldots * (1, 0, 0) = (n, 0, 0)$; we cannot reduce to 1, the identity $(0, 0, 0)$, by raising to a power

and although not abelian

There are g and h in $H_3(\mathbb{Z})$, e.g. $g = (1, 0, 0)$, $h = (0, 1, 0)$, for which $[g, h] \neq 1$

is nilpotent of class 2.

For all f, g, h in $H_3(\mathbb{Z})$, $[f, [g, h]] = 1$

E.g: $(0, 0, 1) = [(1, 0, 0), (0, 1, 0)]$ commutes with any f in $H_3(\mathbb{Z})$, $[f, (0, 0, 1)] = 1$. The set $\{(0, 0, a) \mid a \in \mathbb{Z}\}$ is the *centre* of $H_3(\mathbb{Z})$.

For any *h*, a suitable *i* excludes it from

$N = \{(ap^i, bp^i, cp^i) \mid a,b,c \in \mathbb{Z}, p \text{ prime}\}$

which is a normal subgroup

$(x, y, z)^{-1} * (ap^i, bp^i, cp^i) * (x, y, z)$
$= (ap^i, bp^i, (c + ay - xb)p^i) \in N$

whose cosets form a finite *p*-group

$(ap^i, bp^i, cp^i) * (x, y, z)$
$= (ap^i + x, bp^i + y, (c + ay)p^i + z) \notin N$
for $x, y, z \in \{0, \ldots, p^i - 1\}$

Which means, precisely, that we have a residually finite *p*-group

The example above is by no means atypical — in fact, every finitely generated, torsion-free, nilpotent group is isomorphic to a subgroup of a Heisenberg group over the integers. The theorem essentially says that every element in such a group 'lives inside' a finite *p*-group. Equivalently, any such group can be embedded into a direct product of finite *p*-groups.

With this 1957 theorem, Karl Gruenberg showed that the study of an important class of nilpotent groups could be essentially reduced to the study of finite *p*-groups, whose properties had been the subject of deep investigation since Cauchy's work in the 1840s.

Web link: projecteuclid.org/euclid.bams/1183530287

Further reading: *A Course in the Theory of Groups, 2nd ed.* by DJS Robinson, Springer-Verlag, 1998, chapter 5.