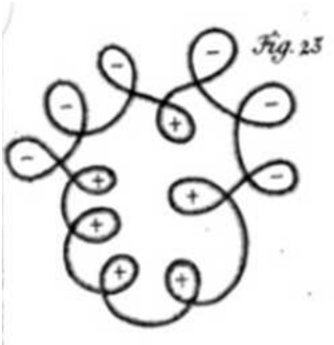
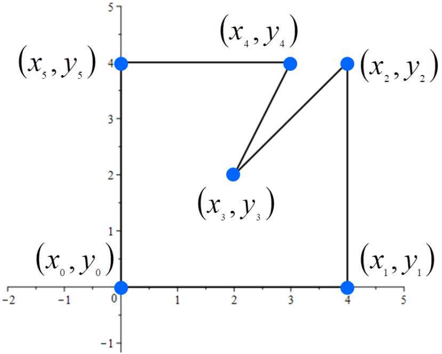
Who Invented the Shoelace Formula?



Robin Whitty, August 2024

The Shoelace Formula

The Shoelace Formula calculates the area of a polygon from the coordinates of its vertices, listed in anticlockwise order.

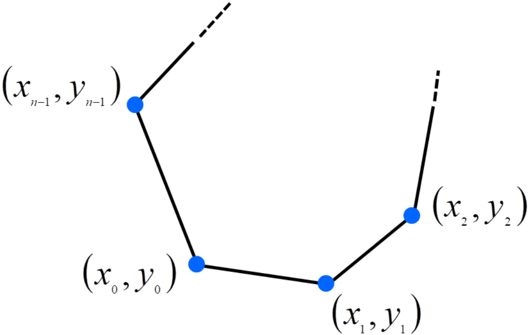


$$\text{Area} = \frac{1}{2}\Big(\; x_0y_1 - y_0x_1 + x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_4 - x_4y_3 + x_4y_5 - x_5y_4 + x_5y_0 - x_0y_5 \;\Big).$$

It takes its name from the ‘interlacing’ of x and y ordinates. But whom might it be named *after*?

General formula

For an *n*-vertex polygon



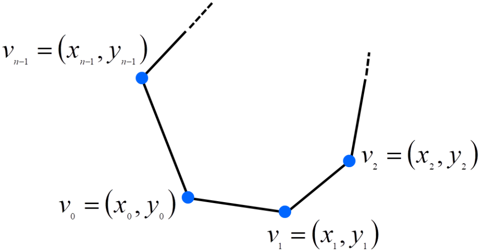
we write

$$\text{Area} = \frac{1}{2}\left(x_0y_1 - x_1y_0 + x_1y_2 - x_2y_1 + \ldots + x_{n-1}y_0 - x_0y_{n-1}\right).$$

It works for polygons that are non-convex and sometimes, but not always, for ones that are non-simple, i.e. with edges crossing other than at vertices.

Using the exterior product

A short-hand simplifies Shoelace-related proofs.



Write v_i for vertex (x_i, y_i). Write $v_i \wedge v_j$, or more simply $v_i v_j$, for the exterior product $x_i y_j - x_j y_i$. Notice that $v_i v_j = -v_j v_i$ and that $v_i^2 = 0$.

Actually, in this 2D context, the exterior product is just the (magnitude of the) cross product of position vectors, calculated as for example

$$v_i \times v_j = \det \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}.$$

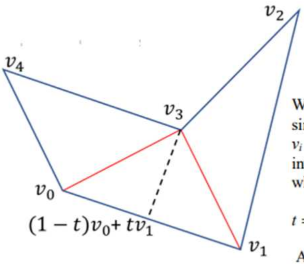
Exterior product version of Shoelace

$$\text{Area} = \frac{1}{2}\left(x_0y_1 - x_1y_0 + x_1y_2 - x_2y_1 + \ldots + x_{n-1}y_0 - x_0y_{n-1}\right).$$

becomes

$$\text{Area} = \frac{1}{2}\left(v_0v_1 + v_1v_2 + \ldots + v_{n-1}v_0\right).$$

Example application:



We apply the Shoelace formula, simplying via $v_i \wedge v_i = 0$ and $v_i \wedge v_j = -v_j \wedge v_i$, just as for the invariants e_i. We get an equation which we may solve for t, giving:

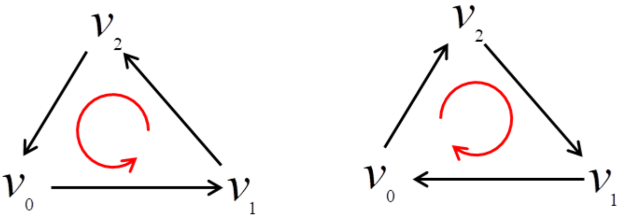
$$t = \frac{v_0v_1 + v_1v_2 + v_2v_3 + v_3v_0 - (v_0v_3 + v_3v_4 + v_4v_0)}{2(v_0v_1 + v_1v_3 + v_3v_0)},$$ omitting the $\wedge$s for greater clarity.

And we recognise three applications of the Shoelace formula! Denote by A_{cl} the polygon area clockwise from of our chosen triangle; by A_{co} the remaining polygon area; and by A_Δ the area of the triangle itself. Then

$$t = \frac{A_{co} - A_{cl}}{2A_\Delta}.$$

Triangular shoelaces

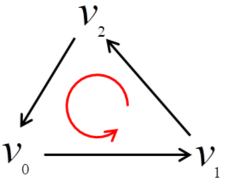
The position vectors of a triangle are given as: v_0, v_1, v_2.



Basic fact: the area of the triangle is the cross product of the direction vectors of any two sides *in anticlockwise order*.

$$
\begin{aligned}
\text{Area} &= \frac{1}{2}(-v_0+v_1)\times(-v_1+v_2) = -\frac{1}{2}(-v_2+v_1)\times(-v_1+v_0) \\
&= \frac{1}{2}(-v_1+v_2)\times(-v_2+v_0) = -\frac{1}{2}(-v_0+v_2)\times(-v_2+v_1) \\
&= \frac{1}{2}(-v_2+v_0)\times(-v_0+v_1) = -\frac{1}{2}(-v_1+v_0)\times(-v_0+v_2)
\end{aligned}
$$

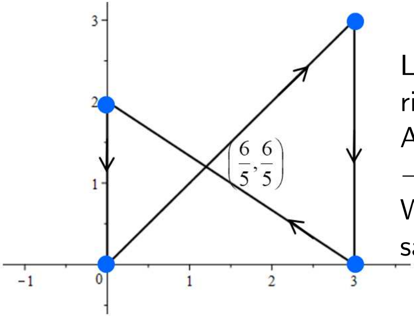
Anticlockwise shoelace



$$
\begin{aligned}
\text{Area} &= \frac{1}{2}(-v_0 + v_1) \times (-v_1 + v_2) \\
&= \frac{1}{2}(-v_0 \times -v_1 + -v_0 \times v_2 + v_1 \times -v_1 + v_1 \times v_2) \\
&= \frac{1}{2}(v_0 \times v_1 + v_2 \times v_0 + 0 + v_1 \times v_2) \\
&= \frac{1}{2}(v_0 \times v_1 + v_1 \times v_2 + v_2 \times v_0) \\
&= \frac{1}{2}(v_0 v_1 + v_1 v_2 + v_2 v_0).
\end{aligned}
$$

So Shoelace follows 'from first principles' for triangles, respecting a 'rule of signs' for clockwise/anticlockwise orientation.

Non-simple shoelace



Left-hand 'triangle' area $= 6/5$; right-hand 'triangle' area $= -27/10$. Area of polygon $= 6/5 - 27/10 = -3/2$.
We shall see that Shoelace gives the same answer in this simple case.

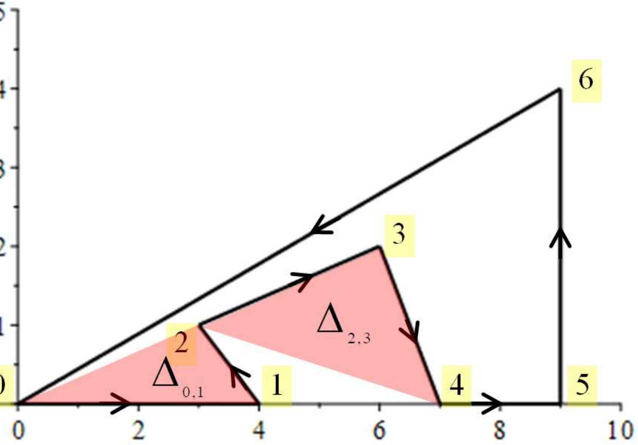
Shoelace:

$$\text{Area } = \frac{1}{2}\left(x_0y_1 - x_1y_0 + x_1y_2 - x_2y_1 + \ldots + x_{n-1}y_0 - x_0y_{n-1}\right).$$

$$\frac{1}{2}\left(0 \times 3 - 3 \times 0 + 3 \times 0 - 3 \times 3 + 3 \times 2 - 0 \times 0 + 0 \times 0 - 0 \times 2\right) = -\frac{3}{2}.$$

Δ, the triangle areas matrix

Denote by Δ_{ij} the area of the triangle on polygon vertices $i, j, j+1$, the numbering taken modulo n. This area is taken as positive or negative according to whether $i, j, j+1, i$ has counterclockwise or clockwise orientation relative to the orientation of the polygon.

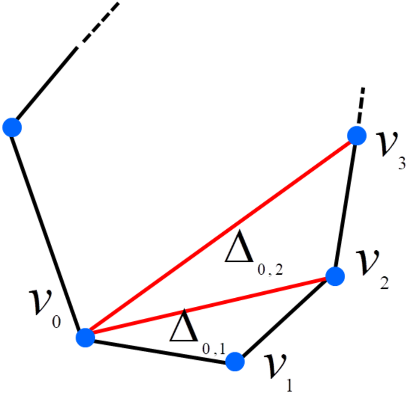


Left we have highlighted areas $\Delta_{0,1} = 2$ and $\Delta_{2,3} = -7/2$.

Note also $\Delta_{0,2} = \Delta_{0,4} = 0$.

Shoelace and the triangle areas matrix

Apply Shoelace to row zero (wlog) of the Δ matrix to calculate all triangle areas from vertex v_0:



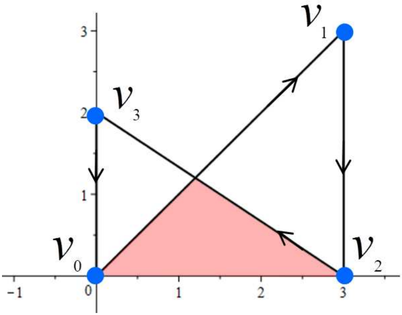
	$\Delta_{0,1}$		$\Delta_{0,2}$		$\cdots$
	$v_0 v_1$		$\cancel{v_0 v_2}$		
$\frac{1}{2}$	$v_1 v_2$	$+$	$v_2 v_3$	$+$	$\cdots$
	$\cancel{v_2 v_0}$		$\cancel{v_3 v_0}$		

Cancellation due to $v_{i,0} = -v_{0,i}$ means that each Δ row sum exactly duplicates Shoelace:

$$\text{Area} \; = \frac{1}{2}\left(v_0 v_1 + v_1 v_2 + v_2 v_3 + \ldots\right.$$

Δ for non-simple polygons

The Δ matrix for the example non-simple polygon is shown below:

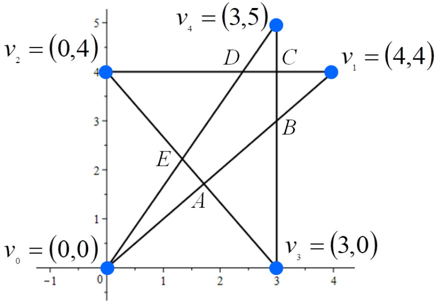


$$\Delta := \begin{bmatrix} 0 & -\frac{9}{2} & 3 & 0 \\ 0 & 0 & -\frac{9}{2} & 3 \\ -\frac{9}{2} & 0 & 0 & 3 \\ 3 & -\frac{9}{2} & 0 & 0 \end{bmatrix}$$

The Shoelace calculation is constructed from just two entries. In the first row these are $\Delta_{0,1} = -9/2$ and $\Delta_{0,2} = 3$.
They are not individually duplicating the components of Shoelace: they add and then subtract the shaded triangle area of 9/5.
Note that $\Delta_{0,1} + 9/5 = -27/10$ while $\Delta_{0,2} - 9/5 = 6/5$, that is, the areas of the two triangles constituting the polygon.

Shoelace may fail for non-simple polygons

The figure below is a 5-vertex pentagram:



The points of intersection of the five sides are

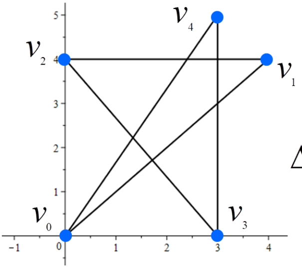
A	$(12/7, 12/7)$
B	$(3, 3)$
C	$(3, 4)$
D	$(12/5, 4)$
E	$(4/3, 20/9)$

The area of the polygon, considered as the area enclosed by the outside sequence of points, $v_0Av_3Bv_1Cv_4Dv_2Ev_0$, is $794/105$.

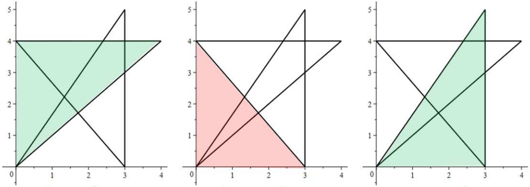
However, Shoelace, applied to the vertices taken in (anticlockwise) order, is $19/2$.

Δ matrix values for the pentagram

Using the Δ matrix it is easy to see what has gone wrong:



$$\Delta := \begin{bmatrix} 0 & 8 & -6 & \frac{15}{2} & 0 \\ 0 & 0 & 8 & -\frac{5}{2} & 4 \\ 8 & 0 & 0 & \frac{15}{2} & -6 \\ -6 & 8 & 0 & 0 & \frac{15}{2} \\ 4 & -2 & \frac{15}{2} & 0 & 0 \end{bmatrix}$$



$\Delta_{0,1} = 8$ $\Delta_{0,2} = -6$ $\Delta_{0,3} = 15/2$

Δ areas shaded, green for positive, red for negative.

$\Delta_{0,1} + \Delta_{0,2} + \Delta_{0,3} = 19/2$

We see that the central pentagonal area has been counted twice!

So whose is the Shoelace formula?

English Wikipedia:

> The formula was described by Albrecht Ludwig Friedrich Meister (1724–1788) in 1769[4] and is based on the trapezoid formula which was described by Carl Friedrich Gauss and C.G.J. Jacobi.[5] The triangle form of the area formula can be considered to be a special case of Green's theorem.
>
> The area formula can also be applied to self-overlapping polygons since the meaning of area is still clear even though self-overlapping polygons are not generally simple.[6] Furthermore, a self-overlapping polygon can have multiple "interpretations" but the Shoelace formula can be used to show that the polygon's area is the same regardless of the interpretation.[7]

Albrecht Meister 1770:

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M. ALBERT. LVDOV. FRID. MEISTER

GENERALIA DE GENESI FIGVRARVM PLANARVM,

ET

INDE PENDENTIBVS EARVM AFFECTIONIBVS

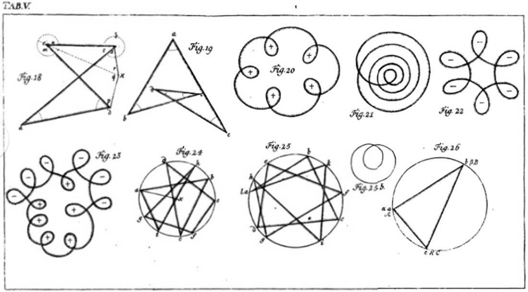
D. VI. IAN. CIƆ IƆCCLXX.

PRAEFATIO.

Ideam figurae planae, ſeu quis petierit à limitibus planum undique terminantibus, ſive a motu lineae in aliquo plano; facile deprehendet, multo latius eandem patere, quam quae a ſola naturalium corporum contemplatio-

The origins of the polygon

Pictures from Meister 1770:



Meister vs. Poinsot:

Beitr Algebra Geom (2012) 53:57–71
DOI 10.1007/s13366-011-0047-5

ORIGINAL PAPER

Polygons: Meister was right and Poinsot was wrong but prevailed

Branko Grünbaum

Abstract The definitions of the term "polygon" as given and used by Meister (1724–1788) in 1770 and by Poinsot (1777–1859) in 1810 are discussed. Since it is accepted that mathematicians are free to define concepts whichever way they like, the claim that one of them is right and the other wrong may appear strange. The following pages should justify the assertion of the title by pointing out some of the errors and inconsistencies in Poinsot's work, and—more importantly—show the undesirable and harmful consequences resulting from it.

Meister's concerns

Google translate from Latin:

Meister, 1770

X. Perimeter changes while maintaining the shape of the area

The configurations can be changed in number of ways. they say the truth among those who depend on a certain and general norm. For example, if he did the perimeter must be changed, with all the angles remaining in the same angles parallel to each other; or by changing the position of each corner in a straight line, parallel to the straight line, placed at the nearest angles on both sides.

(1)

Fig. 39 (2) I move on to the other method. Let ab and bc be connected to the sides of any figure; ga, gb, and gc are diagonals taken at the point g as desired; The triangles agb and cgb will represent a part of the area insofar as it hangs on the sides ab,bc; the remaining area will be shown by the triangles described on the other sides at the same vertex g. It makes no difference whatsoever that the sides ab, bc succeed in the place of the sides, but this change only refers to the triangles agb, gbc. But if, therefore, the sides aB, Bc succeed to the sides ab, bc, of which the triangles agB and Bgc, we must follow the sum of the triangles agb, bgc; it is evident that the value will be the same whether the sides abc or the other side terminated the figure on this side. But this will happen when the points B and B are in a straight line to the right and parallel to the line. For then the triangles abc, aBc are equal. But this value is shown by the sum of the triangles agb+bgc-agc, the latter by the sum of the triangles agB+Bgc-agc, or - agB+Bgc-agc, for the varying position of the triangle

Gauss's disclaimer

Google translate from German:

Gauss letter to Oblers
1825

Ich hätte auch die Lehre von dem Flächeninhalt der Figuren über haupt nennen können, die ich gleichfalls seit 30 und mehren Jahren aus einem von mir bisher für neu gehaltenen Gesichtspunkt betrachtet habe. Dieses letztere ist aber zum Theil ein Irrthm: in der That habe ich erst vor kurzem eine Abhandlung von MEISTER (einem meiner Meinung nach sehr genialen Kopf) im ersten Bander der Novi Commentarii Gottin. kennen gelernt, worin die sache fast ganz auf gleiche Art betrachtete und sehr schön entwickelt wird.

I could also have mentioned the theory of the area of the figures, which I have also been looking at for thirty or more years from a point of view that I have hitherto considered to be new. The latter, however, is partly a mistake: in fact, only recently did I have a treatise by MEISTER (in my opinion a very brilliant mind) in the first volume of the Novi Commentarii Gotting. got to know, in which the matter is viewed in almost exactly the same way and is developed very nicely.

Back to Gauss

Leçons de statique graphique
by Antonio Favaro

Publication date 1885

p.119:

I. — Principe des signes appliqué aux aires (¹).

81. Un point qui se meut d'un mouvement continu dans un plan décrit un *circuit* ou *périmètre* fermé lorsque sa dernière position vient coïncider avec sa position initiale.

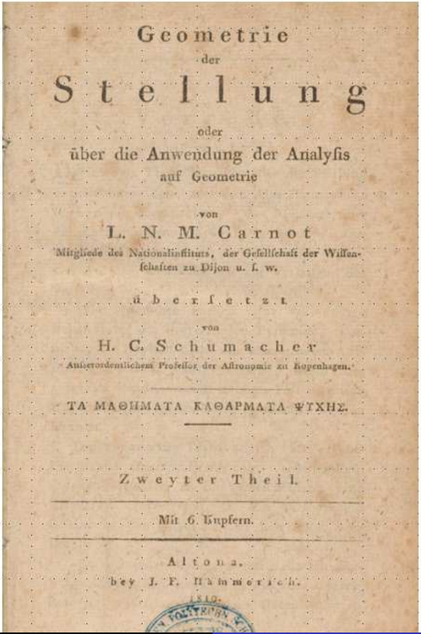
(¹) La règle des signes appliquée aux aires a été établie pour la première fois, dans toute sa généralité, par Möbius (*Der barycentrische Calcul*, etc., Leipzig, 1827, § 17, 18 et *passim*. — *Lehrbuch der Statik*, Leipzig, 1837, § 34). On trouve, toutefois, d'intéressants matériaux pour l'étude de cette question dans un Mémoire de Monge (XVᵉ Cahier du *Journal de l'École Polytechnique*, p. 68). Les premières notions sur les signes des aires, spécialement dans les figures à périmètres croisés, on été données par Meister (*Generalia de genesi figurarum planarum et independentibus earum affectionibus*, dans les *Novi Commentarii Societatis regiæ Scientiarum Gottingensis*, t. I, ad n. MDCCLXIX et MDCCLXX, p. 150). — Lexell (*Solutio problematis geometrici ex doctrina sphæricorum*, dans les *Acta Academiæ Scientiarum imperialis Petropolitanæ*, 1781, I, p. 126 et suiv.) et L'Huillier (*De relatione mutua capacitatis et terminorum figurarum Geometriæ considerata*, Varsoviæ, 1782, p. 23) ont aussi jeté quelque lumière sur ce sujet.

C'est ici le cas de rappeler la célèbre formule de Gauss, au moyen de laquelle, connaissant les coordonnées des sommets $x, y; x', y'; x'', y''; \ldots, x^{n-1}, y^{n-1}$, on peut calculer l'aire de tous les polygones déterminés par l'équation

$$A = \tfrac{1}{2}[x(y' - y^{n-1}) + x'(y'' - y) + x''(y''' - y') + \ldots + x^{n-1}(y - y^{n-2})].$$

Cette formule a été publiée pour la première fois dans la traduction allemande de la *Géométrie de position* de Carnot (*Geometrie der Stellung oder über die Anwendung der Analysis auf Geometrie*, deutsch v. Schumacher, 2. Theil, Altona, 1810, p. 362). Mais c'est encore dans les travaux de Möbius qu'il faut chercher la première étude complète sur les signes des aires dans les figures à contours croisés (voir, outre les deux Ouvrages cités plus haut, *Theorie der elementaren Verwandtschaft*, dans les *Berichte über die Verhandlungen der Sächsischen Gesellschaft der Wissenschaften*, t. 15, p. 42 et suiv. et *Ueber die Bestimmung des Inhalts eines Polyëders*, dans le même Recueil, t. 17). Consulter sur la même question : *Die Elemente der Mathematik*, von Dʳ Richard Baltzer, II. Bd., dritte verbesserte Auflage, Leipzig, 1870, p. 65-68, et *Vermischte Untersuchungen für Geschichte der mathematischen Wissenschaften*, von Dʳ Siegmund Günther, Leipzig, 1876, p. 1-92.

Gauss according to Carnot?


Geometrie
der
Stellung
oder
über die Anwendung der Analysis
auf Geometrie

von
L. N. M. Carnot
Mitgliede des Nationalinstituts, der Gesellschaft der Wissenschaften zu Dijon u. s. w.

übersetzt

von
H. C. Schumacher
Ausserordentlichem Professor der Astronomie zu Kopenhagen.

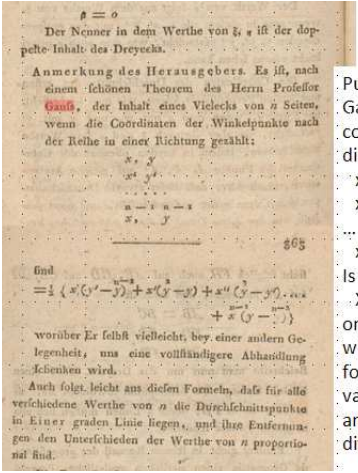
ΤΑ ΜΑΘΗΜΑΤΑ ΚΑΘΑΡΜΑΤΑ ΨΥΧΗΣ.

Zweyter Theil.

Mit 6 Kupfern.

Altona,
bey J. F. Hammerich.
1810.

Or Gauss according to Schumacher?



$\beta = o$

Der Nenner in dem Werthe von ξ, η ist der doppelte Inhalt des Dreyecks.

Anmerkung des Herausgebers. Es ist, nach einem schönen Theorem des Herrn Professor Gauß, der Inhalt eines Vielecks von n Seiten, wenn die Coordinaten der Winkelpunkte nach der Reihe in einer Richtung gezählt:

x, y

x', y'

. . . . .

x^{n-1}, y^{n-1}

565

sind

$$= \tfrac{1}{2}\{x(y' - y^{n-1}) + x'(y^{2} - y) + x''(y^{3} - y') \ldots + x^{n-1}(y - y^{n-2})\}$$

worüber Er selbst vielleicht, bey einer andern Gelegenheit, uns eine vollständigere Abhandlung schenken wird.

Auch folgt leicht aus diesen Formeln, daß für alle verschiedene Werthe von n die Durchschnittspunkte in Einer graden Linie liegen, und ihre Entfernungen den Unterschieden der Werthe von n proportional sind.

Publisher's Note. According to a famous theorem of Gauss, the area of a polygon with n sides, if the coordinates of the vertices are numbered in one direction:

x, y

x', y'

...

x^n-1, y^n-1

Is

XXX

on which He Himself, perhaps, on another occasion, will give us a more complete treatise. It also easily follows from these formulas that for all different values of n the average points lie in a straight line, and their distances are proportional to the differences in the values of n.

Schumacher's formula?

Heinrich Christian Schumacher

19 languages

Article Talk Read Edit View history Tools

From Wikipedia, the free encyclopedia

Not to be confused with Heinrich Christian Friedrich Schumacher.

Prof **Heinrich Christian Schumacher** FRS(For) FRSE (3 September 1780 – 28 December 1850) was a German-Danish astronomer and mathematician.

Heinrich Christian Schumacher

Biography [edit]

Schumacher was born at Bramstedt, in Holstein, near the German/Danish border. He was educated at Altona Gymnasium on the outskirts of Hamburg. He studied in Germany at Kiel, Jena, and Göttingen Universities as well as Copenhagen. He received a doctorate from Dorpat University in Russian Empire in 1807.[1]

From 1808, he was adjunct professor of astronomy in Copenhagen. He directed the Mannheim observatory from 1813 to 1815, and then in 1815 was appointed Professor of Astronomy in Copenhagen and Director of the Observatory.[2]

From 1817 he directed the triangulation of Holstein, to which a few years later was added a complete geodetic survey of Denmark (finished after his death). For the sake of the survey, Schumacher established the Altona Observatory at Altona, and resided there permanently.[2] He cooperated with Carl Friedrich Gauss for the baseline measurement (Braak Base Line) in the village Braak near Hamburg in 1820.

He was elected a Foreign Fellow of the Royal Society of London in 1821, and a Fellow