Who Invented the Shoelace Formula?

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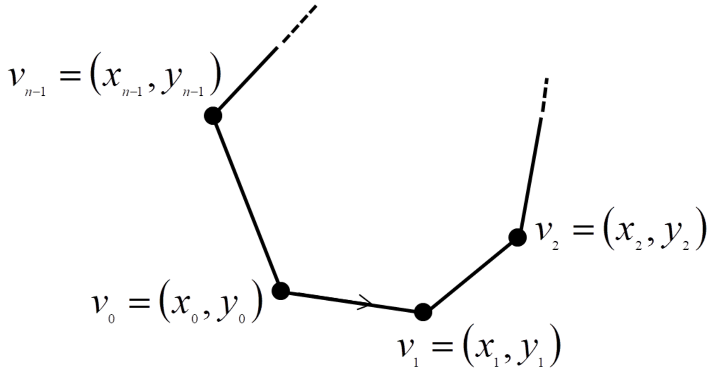
The Shoelace formula allows you to calculate the area of a polygon from the coordinates of its vertices, listed anticlockwise.



$$\text{Area} = \frac{1}{2}(x_0y_1 - x_1y_0 + x_1y_2 - x_2y_1 + \ldots + x_{n-1}y_0 - x_0y_{n-1}) \quad (1)$$

or, more concisely,

$$\text{Area} = \frac{1}{2}(v_0v_1 + v_1v_2 + \ldots + v_{n-1}v_0), \quad (2)$$

where v_iv_j is a 'shorthand' for $x_iy_j - x_jy_i$.

It is surprisingly simple, given that it works for non-convex polygons. It takes its name from the 'interlacing' of x and y ordinates. But whom might it be named *after*?

The formula is often referred to as 'Gauss's' but "the formula was certainly not invented by him" according to the German mathematician Burkard Polster in his excellent Mathologer Youtube video "Gauss's magic shoelace area formula and its calculus companion". A rather cursory trawl through those parts of Gauss's collected works that pertain to plane geometry showed me nothing resembling the above formula. But it is also sometimes called 'the surveyor's area formula' and Gauss, as a loyal subject of the duke of Hanover, undertook the challenge of surveying the duke's territory. Whatever records of that enterprise might exist they have not be examined, even cursorily, by me. In any case Gauss was famously reticent about publishing his discoveries, I suppose to the frustration of even accomplished historians of science.

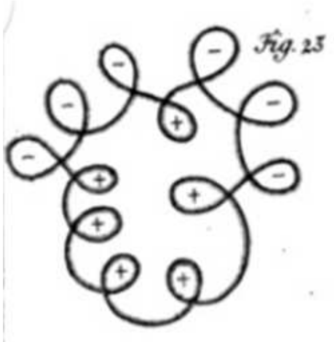
Wikipedia accounts of Shoelace (as I will call it for short) vary greatly from language to language. The English language page is more assertive than most on history "The formula was described by Albrecht Ludwig Friedrich Meister (1724–1788) in 1769 and is based on the trapezoid formula which was described by Carl Friedrich Gauss and C.G.J. Jacobi." Since Gauss was born in 1777 and Jacobi in 1804, the phrase 'based on' is not to be taken literally. Nor, in any case, can I find that Gauss wrote down a 'trapezoid formula' in relation to polygon area. But the work of Meister is definitely relevant.

Albrecht Meister wrote a treatise called *Generalia de genesi figurarum planarum et inde pendentibus earum affectionibus* (dated 1770, in fact, not 1769). This may be regarded as the first systematic account of polygons. It addresses how polygons may be specified, classified and manipulated, and has many pages of diagrams. It talks about area in detail. And Gauss's collected correspondence contains a letter of 1825 to his friend and colleague Wilhelm Olbers in which he says (my cursory translation from German):

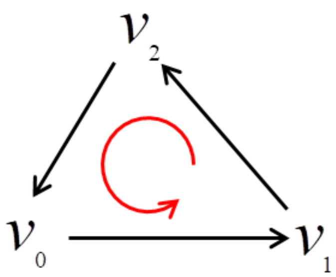
> I could also have mentioned the theory of the area of figures, which I have been looking at for thirty or more years from a point of view that I have hitherto considered to be

new. This, however, is partly a mistake: in fact, only recently did I learn of a treatise by Meister (in my opinion a very brilliant mind) in the first volume of the Novi Commentarii Gotting, in which the matter is viewed in almost exactly the same way and is developed very nicely.

Nevertheless *Generalia de genesi figurarum planarum* definitely does not describe Shoelace, nor indeed any formulae. What was it that so impressed Gauss? I think it was Meister's invention of 'signed area', meaning that area created by a closed curve turning anticlockwise should be taken to be positive while oppositely oriented areas should be negative. He has several diagrams relating to signed area, one of which is reproduced in figure (a) below



(a)



(b)

One supposes that, for a mathematician of Gauss's stature, this 'rule of signs' was the single, essential insight needed for polygon area; everything else was just bookkeeping. He would hardly have thought it worthwhile recording the fact that the area of a triangle, specified by the direction vectors of its three sides, is half the cross product of any two of these vectors, taken anticlockwise. (The terminology is anachronistic but vector algebra was, in some form, already present in eighteenth century in two and three dimensional geometry.) In figure (b), above, v_0, v_1 and v_2 are position vectors (coordinate pairs). The direction vectors of the sides joining v_0 to v_1 and v_1 to v_2 are, respectively, $v_1 - v_0$ and $v_2 - v_1$. So the bookkeeping goes as follows:

$$\begin{aligned}
\text{Area} &= \frac{1}{2}(v_1 - v_0) \times (v_2 - v_1) && \text{half cross product of edges} \\
&= \frac{1}{2}(v_1 \times v_2 + v_1 \times -v_1 - v_0 \times v_2 - v_0 \times -v_1) \\
&= \frac{1}{2}(v_1 \times v_2 + 0 - v_0 \times v_2 + v_0 \times v_1) && \text{since } v_i \times v_i = 0 \\
&= \frac{1}{2}(v_1 \times v_2 + v_2 \times v_0 + v_0 \times v_1) && \text{since } v_i \times v_j = -v_j \times v_i \\
&= \frac{1}{2}(v_0 v_1 + v_1 v_2 + v_2 v_0)\,. && \text{in the notation of equation (2).}
\end{aligned}$$

And Shoelace has emerged directly from the cross product rule; and indeed equation (2) is simply Shoelace in cross product notation.

It is still something of a leap of imagination to see that clockwise and anticlockwise triangles will combine, in tracing a complicated non-convex polygon, to produce its total area. And although I can believe that this was implicit in what Meister wrote, and explicit in what Gauss thought, still we have yet to see it announced as a formula. This announcement happened, we have it on good authority, in 1810. *Lezioni di statica grafica*, an 1877 textbook by the Italian mathematician and historian of science Antonio Favaro (1847–1922), was translated, as *Leçons de statique graphique*, in two parts by a monsieur Paul Terrier. I can discover nothing about monsieur Terrier, except that he studied at the elite Ècole centrale Paris, but he was very thorough in his translation of *Lezioni*, adding copious appendices and footnotes. Chapter 4 of Part 2 opens with a section on "Principe des signes appliqué aux aires". And immediately there is a long footnote in which Terrier traces “Les premières notions sur les signes des aires, spécialement dans les figures aux périmètres croisés” to Meister's

1770 treatise. And he continues “C’est ici le cas de rappeler la célèbre formule de Gauss ...”; and recall the celebrated formula he accordingly does. And he goes on to state (my translation) “This formula was published for the first time in the German translation of *Géométrie de position* de Carnot”.

Lazare Carnot’s *Géométrie de position* was published in 1803 and was published in German as *Geometrie der Stellung* in 1810. As with Favaro’s *Lezioni* the translation provides additional material and this is why Paul Terrier cites it as the Shoelace’s place of origin. And on page 362 we find a footnote marked “Anmerkung des Herausgebers”, meaning “editor’s note”, or in this case, I think, “translator’s note”, which reads

> According to a famous theorem of Gauss, the area of a polygon with n sides, if the coordinates of the vertices are numbered in one direction:
>
> $$\begin{array}{ll} x, & y \\ x', & y' \\ \ldots & \\ x^{n-1}, & y^{n-1} \end{array}$$
>
> is
> $$\frac{1}{2}\left\{x\left(y'-y^{n-1}\right)+x'\left(y^2-y\right)+x''\left(y^3-y'\right)+\ldots+x^{n-1}\left(y-y^{n-2}\right)\right\}.$$
> on which he himself, perhaps, on another occasion, will give us a more complete treatise.

There is no mention of the formula in Carnot’s 1803 text. The translator for the German edition was Heinrich Christian Schumacher (1780–1850), a mathematician and astronomer from the German-Norwegian borders. He studied law at Göttingen and Kiel, graduating in 1806 from the former, where he then took up scientific studies under Gauss. In *Monthly Notices of the Royal Astronomical Society*, Vol. 11, February 1851, pp.73–81 we read

> In 1805 he began his translation of Carnot’s *Géometri de position* into German, *ad recreationem animi*, as he says himself in the introduction to the work. This translation, with notes by Gauss, was published at Hamburg in two volumes.

There is no doubt that Gauss and Schumacher were, and remained, close colleagues. In the complete correspondence of Gauss, which may be read at the website gauss.adw-goe.de, a sixth of the over eight thousand entries are letters to or from Schumacher. Of those predating the publication of *Geometrie der Stellung* none appear to relate to polygon area. In any case, Schumacher remained at Göttingen until 1810, when he was appointed extraordinary professor of astronomy at Copenhagen. We have to assume that Schumacher’s footnote on page 362 of *Geometrie der Stellung* had at least Gauss’s blessing, if it was not actually dictated by Gauss. We can image Schumacher adding the “on which he himself, perhaps, on another occasion, will give us a more complete treatise” to tease Gauss, the reluctant scribe.