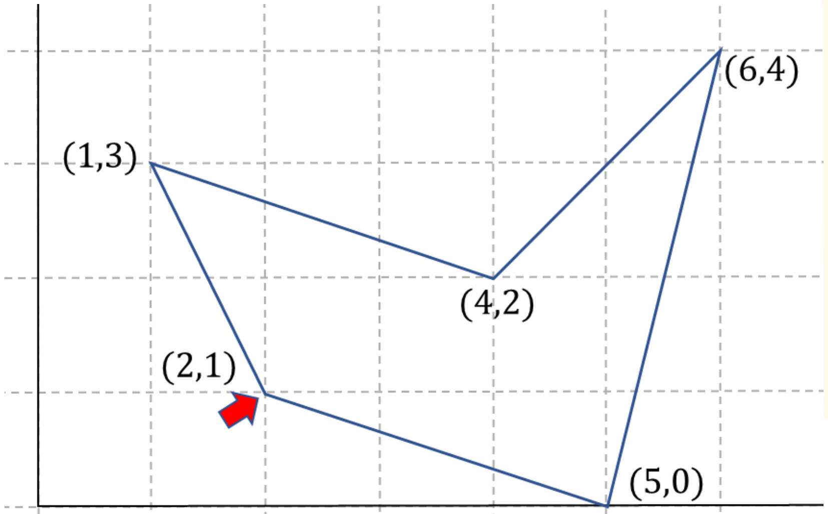
THEOREM OF THE DAY

The Shoelace Formula *Suppose the n vertices of a simple polygon in the Euclidean plane are listed in counterclockwise order as $(x_0, y_0), \ldots, (x_{n-1}, y_{n-1})$. Then the area A of the polygon may be calculated as:*

$$A = \frac{1}{2}\left(x_0y_1 - x_1y_0 + \ldots + x_{n-2}y_{n-1} - x_{n-1}y_{n-2} + x_{n-1}y_0 - x_0y_{n-1}\right).$$



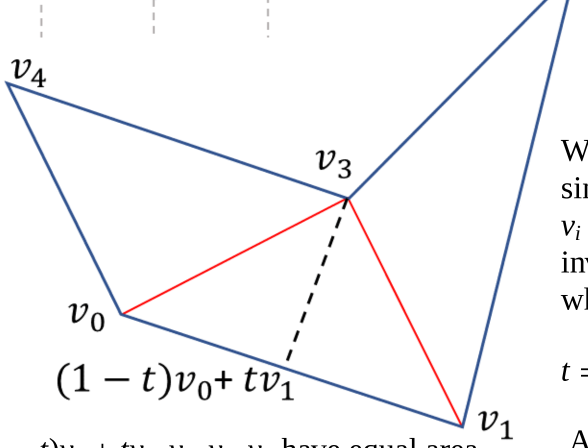
Example
Because the polygon on the left has lattice point vertices, Pick's theorem gives its area as:

$$I + B/2 - 1 = 6 + 6/2 - 1 = 8,$$

where I (resp. B) is number of interior (resp. boundary) lattice points. We can confirm that the Shoelace formula gives the same value, calculating counterclockwise from the arrow:

$$\begin{array}{rcccl} \frac{1}{2}\times(& 2\times 0 & - & 1\times 5 & \\ + & 5\times 4 & - & 0\times 6 & \\ + & 6\times 2 & - & 4\times 4 & \\ + & 4\times 3 & - & 2\times 1 & \\ + & 1\times 1 & - & 3\times 2 &) = 8. \end{array}$$

An Application
We may triangulate a polygon on n vertices by adding $n-3$ diagonals, as illustrated on the right. We would like to test if some straight line joining a triangle vertex to the opposite polygon edge bisects the area of the polygon. In our diagram this requires a value of $t \in [0, 1]$ for which the polygons $v_0, (1-t)v_0 + tv_1, v_3, v_4$ and $(1-t)v_0 + tv_1, v_1, v_2, v_3$ have equal area.



We apply the Shoelace formula, simplying via $v_i \wedge v_i = 0$ and $v_i \wedge v_j = -v_j \wedge v_i$, just as for the invariants e_i. We get an equation which we may solve for t, giving:

$$t = \frac{v_0v_1 + v_1v_2 + v_2v_3 + v_3v_0 - (v_0v_3 + v_3v_4 + v_4v_0)}{2(v_0v_1 + v_1v_3 + v_3v_0)},$$ ommitting the $\wedge$s for greater clarity.

And we recognise three applications of the Shoelace formula! Denote by A_L the polygon area to the left of our chosen triangle; by A_R the remaining polygon area; and by A_Δ the area of the triangle itself. Then

$$t = \frac{A_R - A_L}{2A_\Delta}.$$

For our example polygon, again using Pick, $A_L = 5/2$, $A_R = 8 - 5/2 = 11/2$ and $A_\Delta = 5/2$. This gives $t = (11/2 - 5/2)/5 = 3/5$ (a little to the right of our dotted line).

The Shoelace formula was invented in 1769 by Albrecht Meister, but it is widely attributed to Gauss who made significant discoveries about polygons at the age of 18 in the 1790s. It may now be seen as an application of Green's Theorem (1828).

Using the exterior algebra

- Invariants $e_1, \ldots, e_n$, multiplied and added over a field (e.g. $\mathbb{R}$) to give 'formal' expressions.

  E.g. $E = e_1e_2e_3 - 3e_1e_3e_2 + \sqrt{2}e_3^3$.
- Expressions multiply using the exterior (wedge) product $E \wedge F$ (ommitted for single invariants as in the above example) using the following rule:

  $$e_ie_j = \begin{cases} 0 & i = j \\ -e_je_i & i \neq j \end{cases}$$

  Thus our example expression above simplifies:

  $$E = e_1e_2e_3 + 3e_1e_2e_3 + 0 = 4e_1e_2e_3.$$
- Encoding $v_i = (x_i, y_i)$ as $x_ie_1 + y_ie_2$, we have

  $$v_i \wedge v_j = (x_iy_j - x_jy_i)e_1e_2.$$
- Now we can express the Shoelace formula very concisely:

  $$A = \frac{1}{2}(v_0 \wedge v_1 + \ldots v_{n-1} \wedge v_0)$$

  (or, more precisely, the coefficient of e_1e_2 in this summation).

Web link: www.math.tolaso.com.gr/?p=1451
Further reading: *The Shoelace Book* by Burkard Polster, AMS, 2006.

Created by Robin Whitty for www.theoremoftheday.org